PROBABILITY

- Concepts

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1

Probability

Decisions under uncertainty

- What are the chances that sales will decrease if we increase prices?
- What is the likelihood a new assembly method will increase productivity?
- How likely is it that the project will be finished on time?
- What is the chance that a new investment will be profitable?

2

Probability

- A numerical measure of the likelihood of an event to occur
- Language of uncertainty
- Probabilities can be used as measures of the degree of uncertainty of an event (the 4 listed before)
- Values range from 0 to 1
- Near zero – unlikely occurrence of an event
- Near one – an event is almost certain to occur

Event

- One or more of the possible outcomes of doing something (can be a combination of outcomes)
- Tossing a Coin – Head, Tail
- Drawing from a deck of cards – ace of spades
- Being picked from a class of 100 to answer a question

Experiment

- The activity that produces such an event is called experiment
- Characteristics: well-defined outcomes, on any trial one and only one outcome will occur, by chance or random variability
- Coin-toss experiment: head or tail with equal chances (no chances of landing on its edge, etc.)
- Probability of the event Head = probability of the event Tail = 0.5

Sample Space

- An experimental outcome is called a 'Sample Point'
- The set of all possible outcomes (or sample points) constitute the 'Sample Space'
- Coin-toss experiment: S = {Head, Tail}
- Card-drawing experiment: S = {…} – 52 members – spade, heart, diamond, club; 1, 2,3,.., 10,Jack,Queen,King
- Rolling a dice: S = {1,2,3,4,5,6}

Mutually Exclusive, Collectively Exhaustive Events – mece

- If one and only one of the events can take place at a time, the events are mutually exclusive
- Coin-toss: On any toss (or trial), either heads or tails may turn up, but not both
- If one event occurs, then the other will not occur (the events have no sample points in common)

Mutually Exclusive, Collectively Exhaustive Events – mece

- When the list of possible outcomes of an experiment includes every possible outcome, the list of events is said to be collectively exhaustive
- Coin-toss: ‘Head’ and ‘Tail’ are collectively exhaustive (unless the coin stands on its edge, etc.)

Probability – Notation

- Probability of an event 'A' happening is denoted by

$$P(A)$$

- This is called marginal probability or unconditional probability

$$P('Head') = 0.5$$

Counting Rules

Identifying possible outcomes of an Experiment

Counting Rules – Multiple-step experiments

- If an experiment has ‘k’ steps with
  - n_1 possible outcomes in the first step
  - n_2 possible outcomes in the second step
  - And so on
- Then, the total number of experimental outcomes is given by

$$(n_1).(n_2).\cdots(n_k)$$

Counting Rules – Multiple-step experiments

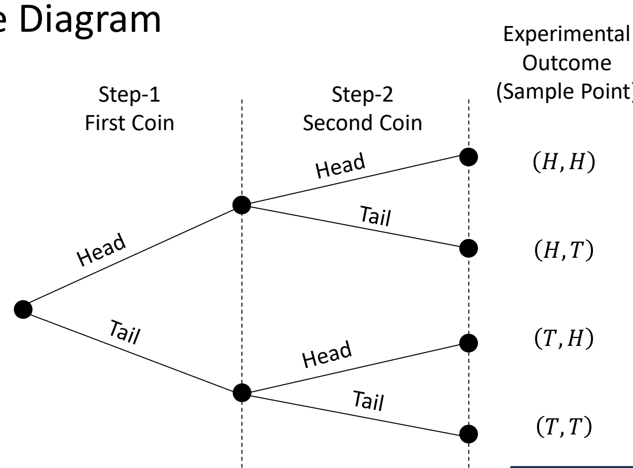
- Tossing 2 coins:
- Step-1: Toss the 1st Coin – 2 outcomes
- Step-2: Toss the 2nd Coin – 2 outcomes

- How many possible outcomes?
- The total number of experimental outcomes

$$(n_1).(n_2) = 2 \times 2 = 4$$

Counting Rules – Multiple-step experiments

- Tree Diagram



$S = \{(H,H),(H,T),(T,H),(T,T)\}$

mece

Counting Rules – KP&L

- Kentucky Power & Light Company – planning for a Capacity expansion project
- 2 steps: Design, Construction
- Time to complete the Project is of interest (Goal – 10 months)
- An analysis of similar construction projects revealed possible completion times – 2, 3, or 4 months for Design; 6, 7, or 8 months for construction
- Possible no. of outcomes is $3 \times 3 = 9$

Counting Rules – KP&L

Stage 1 Design	Stage 2 Construction	Experimental Outcome
2	6	(2, 6)
2	7	(2, 7)
2	8	(2, 8)
3	6	(3, 6)
3	7	(3, 7)
3	8	(3, 8)
4	6	(4, 6)
4	7	(4, 7)
4	8	(4, 8)

Counting Rules – Combinations

- The experiment involves selecting ‘n’ out of ‘N’ objects

$$C_n^N = \binom{N}{n} = NC_n = \frac{N!}{(N-n)!\,n!}$$

- In a quality control procedure 2 out of 5 are selected randomly to test for defects

Counting Rules – Combinations

- In a quality control procedure 2 out of 5 are selected randomly to test for defects
- If the 5 parts are A, B, C, D, and E; then
- Possible no. of outcomes is

$$5C_2 = \frac{5!}{(5-2)!\,2!} = \frac{(1 \times 2 \times 3 \times 4 \times 5)}{(1 \times 2 \times 3) \times (1 \times 2)}$$
$$= 10$$

$$S = \{AB, AC, AD, AE, BC, BD, BE, CD, CE, DE\}$$

Counting Rules – Permutations

- The experiment may involve selecting ‘n’ out of ‘N’ objects; but the ‘ORDER’ is important

$$P_n^N = n!\binom{N}{n} = n!\,NC_n = \frac{N!}{(N-n)!}$$

- In a quality control procedure 2 out of 5 are selected randomly to test for defects

Counting Rules – Permutations

- In a quality control procedure 2 out of 5 are selected randomly to test for defects
- If the 5 parts are A, B, C, D, and E; then
- Possible no. of outcomes is

$$5P_2 = 2!\frac{5!}{(5-2)!\,2!}$$
$$= (1 \times 2) \times \frac{(1 \times 2 \times 3 \times 4 \times 5)}{(1 \times 2 \times 3) \times (1 \times 2)} = 20$$

$$S = \left\{ \begin{matrix} AB, BA, AC, CA, AD, DA, AE, EA, BC, \\ CB, BD, DB, BE, EB, CD, DC, CE, EC, DE, ED \end{matrix} \right\}$$

Assigning Probabilities

Assigning Probabilities

- Two basic requirements MUST be met:
- A_i is the i[th] experimental outcome (Sample Point)

$$0 \leq P(A_i) \leq 1$$

$$P(A_1) + P(A_2) + \cdots + P(A_n) = 1$$

Assigning Probabilities

- THREE approaches:
  - Classical
  - Relative Frequency
  - Subjective

Assigning Probabilities – Classical

- Appropriate when there is equiprobable outcome
- Tossing a fair coin – Head and Tail – are equally likely
- $P(H) = 0.5, P(T) = 0.5$
- Rolling a dice – 1, 2, 3, 4, 5, and 6 – are equally likely
- $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$

Assigning Probabilities – Classical

$$P(A_i) = \frac{1}{n}$$

- In both the experiments, the two requirements are satisfied
- Even before conducting the experiments, probabilities can be computed

Assigning Probabilities – Rel. Frequency

- Q. What is the probability of rain today?
  a) 24th Sept – last 50 years take data – $n/50$
  b) Take rain fall data in the last 10,000 days

- We may not be able to answer without conducting an experiment

Assigning Probabilities – Rel. Frequency

- Insurance Industry, etc.
- Suppose an insurance company knows from the past actuarial data that ‘of all males 40 years old, about 60 out of every 100,000 will die within a 1-year period
- Probability of death for that age group is $\frac{60}{100{,}000} = 0.0006$
- More trials, greater accuracy (but at the expense of time and cost)

Assigning Probabilities – Rel. Frequency

- Consider a study of waiting times in the X-ray department in a hospital

Number Waiting A_i	Number of Days Outcome Occurred	Assigned Probability $P(A_i)$
0	2	2/20 = 0.1
1	5	0.25
2	6	0.30
3	4	0.20
4	3	0.15
	20	1.00

Assigning Probabilities – Subjective

- System's Law of Change – as the probability of NO accidents tending to 'zero', the possibility of accidents will increase
- Based on experience or intuition (degree of belief) – suppose A&B offer to purchase a house
- A believes that their offer will be accepted with 0.8 probability; However, B believes that their offer will be accepted with 0.6 probability

Assigning Probabilities – KP&L

Experimental Outcome (Sample Point)	No. of Projects having these Times	Assigned Probability $P(A_i)$
(2, 6)	6	6/40 = 0.15
(2, 7)	6	0.15
(2, 8)	2	0.05
(3, 6)	4	0.10
(3, 7)	8	0.20
(3, 8)	2	0.05
(4, 6)	2	0.05
(4, 7)	4	0.10
(4, 8)	6	0.15
	40	1.00

Events and their Probabilities

Events and Their Probabilities

- Sample points and events provide the foundation for the study of Probability
- Event: A collection of Sample Points
- KP&L – Project Manager is interested in the event that the entire project can be completed in 10 months or less
- $C = \{(2,6), (2,7), (2,8), (3,6), (3,7), (4,6)\}$
- Event C is said to occur if any one of these 6 sample points appear as the outcome

Events and Their Probabilities

- Event that the entire project is completed in less than 10 months
- $L = \{(2,6), (2,7), (3,6)\}$
- Event that the entire project is completed in more than 10 months
- $M = \{(3,8), (4,7), (4,8)\}$
- We have computed the probabilities of sample points

Events and Their Probabilities

- $P(C) = P(2,6) + P(2,7) + P(2,8) + P(3,6) + P(3,7) + P(4,6) = 0.15 + 0.15 + 0.05 + 0.10 + 0.20 + 0.05 = 0.70$

- $P(L) = P(2,6) + P(2,7) + P(3,6) = 0.15 + 0.15 + 0.10 = 0.40$

- $P(M) = P(3,8) + P(4,7) + P(4,8) = 0.05 + 0.10 + 0.15 = 0.30$

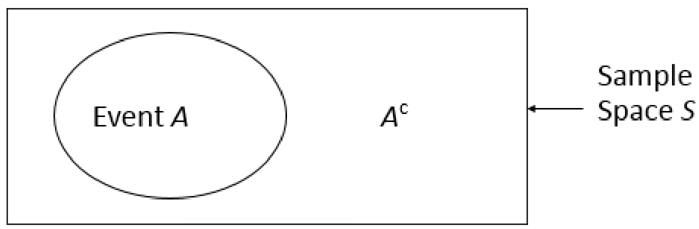
Probability Relationships

– Law of Addition

– Law of Multiplicity

Probability Relationships

- Complement of an event
- A – an event
- A^c – the complement of A – the event consisting of all sample points that are not in A



Venn Diagram

35

Probability Relationships

- A sales manager, after reviewing sales reports, states that 80% of new customer contacts result in NO SALE
- A – event of a SALE
- A^c – event of NO SALE

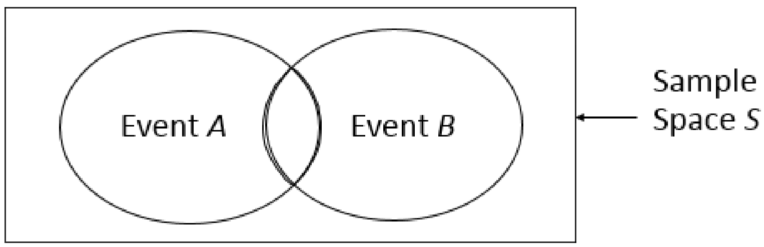
- $P(A) = 1 - 0.80 = 0.20;$
- $P(A^c) = 1 - P(A) = 1 - 0.20 = 0.80$

$$P(A) + P(A^c) = 1$$

36

Union of two events

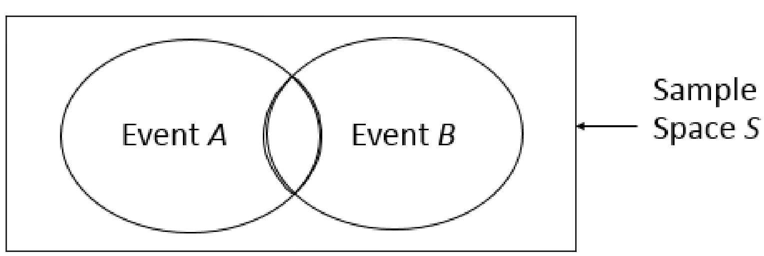
- The union of events A and B is the event containing all sample points that are in A and B or both. The union of events A and B is denoted by A ∪ B. Termed as $P(A \; or \; B)$



Venn Diagram

Intersection of two events

- The intersection of events *A* and *B* is the set of all sample points that are in both *A* and *B.* The union of events A and B is denoted by A ∩ B. Termed as $P(A \; and \; B)$ - also, called Joint Probability



Venn Diagram

Law of Addition

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Consider an assembly plant with 50 employees
- On time completion, no defect product – criteria for performance evaluation
- At the end of a performance evaluation period
  - 5 of the 50: completed work late
  - 6 of the 50: assembled a defective product
  - 2 of the 50: completed work late and assembled a defective product

Law of Addition

- L – event that the work is completed late
- D – event that the assembled product is defective
- $P(L \cup D) = P(L) + P(D) - P(L \cap D) = 0.10 + 0.12 - 0.04 = 0.18$
- There is 0.18 probability that a randomly selected employee received a poor performance rating

Law of Addition

- A software company – Personnel Manager conducted a study
- 30% of the employees who left the Company within 2 years did so primarily because of dissatisfaction with salary
- 20% left because they were dissatisfied with their work assignments
- 12% both
- Q. what is the probability that an employee who leaves within 2 years does so because of dissatisfaction with salary, dissatisfaction with the work assignment, or both?

41

Law of Addition

- Mutually exclusive events:
- $P(A \cap B) = 0$
- $P(A \cup B) = P(A) + P(B)$
- Tossing a coin

42

Conditional Probability

- Often, the probability of an event is influenced by whether a related event already occurred
- Suppose we have an event A with $P(A)$
- If we obtain a new information that a related event B already occurred – then, we will revise the probability of event A – this is called 'conditional probability'
- Probability of 'A' given 'B', denoted by $P(A|B)$

Conditional Probability

- Consider the promotion status of male and female officers in a Police Force

	Men	**Women**	**Total**
Promoted	288	36	324
Not Promoted	672	204	876
Total	960	240	1200

- A committee of female officers raised a discrimination case on the basis that '288 male officers had received promotions, but only 36 female officers had received promotions

Conditional Probability

- Administration argued that 'the relatively low number of promotions for female officers was not due to discriminations, but due to the fact that relatively few female officers are in the force'
- M – event that an officer is a man
- W – event that an officer is a woman
- A – event that an officer is promoted
- A^c – event that an officer is not promoted

Conditional Probability

- Joint Probabilities and Marginal Probabilities

	Men	Women	Total
Promoted	288	36	324
Not Promoted	672	204	876
Total	960	240	1200

	Men	Women	Total
Promoted	0.24	0.03	0.27
Not Promoted	0.56	0.17	0.73
Total	0.80	0.20	1.00

Conditional Probability

- Marginal Probabilities (Unconditional Probabilities) – appear in the margins of the Joint Probability Table
- $P(M) = \frac{960}{1200} = 0.80$
- $P(W) = \frac{240}{1200} = 0.20$
- $P(A) = \frac{324}{1200} = 0.27$
- $P(M) = \frac{876}{1200} = 0.73$

Conditional Probability

- Joint Probabilities (intersection of two events)
- $P(M \cap A) = \frac{288}{1200} = 0.24$
- $P(M \cap A^c) = \frac{672}{1200} = 0.56$
- $P(W \cap A) = \frac{36}{1200} = 0.03$
- $P(W \cap A^c) = \frac{204}{1200} = 0.17$
- Note that $P(A) = P(M \cap A) + P(W \cap A)$

Conditional Probability

- Conditional Probabilities
- Probability that an officer is promoted given that the officer is a man?
- The event M has already occurred. So, concerned with 960 only.
- $P(A|M) = \frac{288}{960} = 0.30$
- $P(A|M) = \frac{288/1200}{960/1200} = \frac{P(M \cap A)}{P(M)} = 0.30$

Conditional Probability

- Conditional Probabilities
- Two events A and B (dependent)

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Or

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Conditional Probability

- Conditional Probabilities
- Two events A and B (independent)

$$P(A|B) = P(A)$$

Or

$$P(B|A) = P(B)$$

Law of Multiplicity

- Conditional Probabilities
- Two events A and B (dependent)

$$P(A \cap B) = P(B).P(A|B) = P(A).P(B|A)$$

- Two events A and B (independent)

$$P(A \cap B) = P(B).P(A|B) = P(A).P(B|A)$$
$$= P(A).P(B)$$

Conditional Probability

- A telecommunications company offers services – high-speed internet, cable television, telephone
- For a particular city, it is known that 84% of the households subscribe to high-speed internet service
- H – event that a household subscribes to high-speed internet service

$$P(H) = 0.84$$

Conditional Probability

- It is also known that the probability that a household that already subscribes to high-speed internet service also subscribes to cable television service (say, event ‘C’) is 0.75

$$P(C|H) = 0.75$$

Conditional Probability

- Q. What is the probability that a household that already subscribes to both high-speed internet service and cable television service?

$$P(H) = 0.84;\ P(C|H) = 0.75$$

$$P(C \cap H) = P(C).P(H|C) = P(H).P(C|H)$$

$$P(C \cap H) = P(H).P(C|H) = 0.84 \times 0.75$$

$$= 0.63$$

Independent Events

- Petrol Bunk – 80% customers use credit card
- Q. What is the probability that the next two customers use a credit card?
- A – 1st customer uses credit card
- B – 2nd customer uses credit card
- Since, no information, assume independent events
- Ans: 0.64

Bayes' Theorem

- The basic formula for conditional probability under dependence

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Or

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

- is called Bayes' Theorem

Bayes' Theorem

$$\boldsymbol{P(A \cap B) = P(B).P(A|B) = P(A).P(B|A)}$$

- The origin of the concept of obtaining posterior probabilities with limited information is attributable to the "Reverend Thomas Bayes (1702-1761)"

Bayes' Theorem

- A manufacturing firm receives shipments of parts from two different suppliers
- A_1 - event that a part is from supplier 1
- A_2 - event that a part is from supplier 2
- Currently, 65% of the parts purchased are from supplier 1 and 35% are from supplier 2

$$P(A_1) = 0.65$$
$$P(A_2) = 0.35$$

Bayes' Theorem

- Quality Levels:

	% of Good Parts	% of Bad Parts
Supplier 1	98	2
Supplier 2	95	5

- $P(G|A_1) = 0.98$
- $P(B|A_1) = 0.02$
- $P(G|A_2) = 0.95$
- $P(B|A_2) = 0.05$

Bayes' Theorem

- Suppose that the parts are used in the process and a machine breaks down because it attempts to process a bad part.
- Q. What is the probability that it came from supplier 1?

$$P(A_1|B)$$

Bayes' Theorem

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)}$$

- $P(A_1 \cap B) = P(A_1).P(B|A_1) = 0.65 \times 0.02 = 0.0130$
- $P(B) = P(A_1 \cap B) + P(A_2 \cap B)$
- $P(A_2 \cap B) = P(A_2).P(B|A_2) = 0.35 \times 0.05 = 0.0175$
- $P(B) = P(A_1 \cap B) + P(A_2 \cap B) = 0.0130 + 0.0175 = 0.0305$
- $P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{0.0130}{0.0305} = 0.4262$

Bayes' Theorem

- $P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{0.0130}{0.0305} = 0.4262$

- We started with $P(A_1) = 0.65$. but, given the information that the part is bad, the probability that the part is from supplier 1 drops to 0.4262

Bayes' Theorem for mece

Posterior

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_n)P(B|A_n)}$$

Prior (unconditional)

Conditional

References

- Anderson, D.R., D.J. Sweeney, T.A. Williams, J.D. Camm, J.J. Cochran, M.J. Fry, and J.W. Ohlmann, Statistics for Business and Economics, 14th edition, Cengage Learning
- Levin, R.I., and D.S. Rubin, Statistics for Management, 7th edition, PHI