CONTINUOUS PROBABILITY DISTRIBUTIONS

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Continuous Probability Distributions

Introduction

- The Industrial Chemicals Division of P&G – fatty alcohol – coconut oil & petroleum-based derivatives
- $x =$ Coconut oil price per pound of fatty alcohol
- $y =$ Petroleum raw material price per pound of fatty alcohol
- $d = x - y$: key to profitability of fatty alcohol production

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Continuous Probability Distributions

d	Probability, $f(d)$
≤ \$0.065	0.90
≤ \$0.035	0.50
≤ \$0.0045	0.10

x	Probability, $f(x)$
Discrete	Prob of x assuming a particular value
Continuous	Prob of x assuming any value in an interval

Continuous Probability Distributions

- Area under the graph $f(x)$ is 'zero' at a particular value of x
- Hence, the probability that the random variable takes a particular value is 'zero'

- Normal distribution – wide applicability, extensive use in statistical inference
- Also, Uniform and Exponential distributions will be discussed

Uniform Probability Distribution

- Flight time of an airplane from A to B – say, 120 to 140 minutes
- From data, probability of a flight time in any 1-minute interval is the same as the probability of a flight time within any other 1-minute interval (in the interval 120-140)

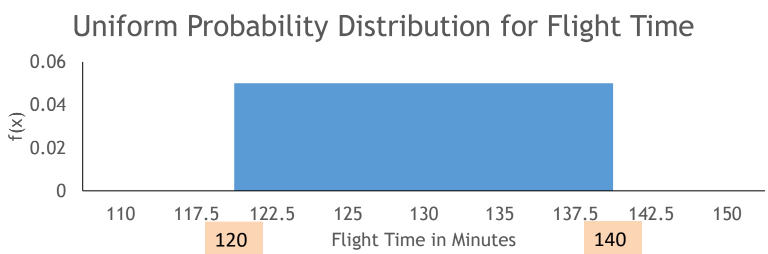
$$f(x) = \begin{cases} \frac{1}{20} & for\ 120 \leq x \leq 140 \\ 0 & oterwise \end{cases}$$

Uniform Probability Distribution

$$f(x) = \begin{cases} \frac{1}{(b-a)} & for\ a \leq x \leq b \\ 0 & otherwise \end{cases}$$

- Q. what is the probability that the flight time is between 120 and 130 minutes?
- What is P$(120 \leq x \leq 130)$?
- Ans: 0.5

Uniform Probability Distribution



$$f(x) = \begin{cases} \frac{1}{20} & for\ 120 \leq x \leq 140 \\ 0 & otherwise \end{cases}$$

- Area under the graph from 120 to 130

$$= width \times height = 10 \times \frac{1}{20} = 0.5$$

Uniform Probability Distribution

- Area under the graph from 126 to 134

$$= width \times height = 8 \times \frac{1}{20} = 0.4$$
$$= P(126 \leq x \leq 134)$$

- Area under the graph from 120 to 140

$$= width \times height = 20 \times \frac{1}{20} = 1$$
$$= P(120 \leq x \leq 140)$$

- Thus, probability that x takes a value between a and b is given by the area under the graph $f(x)$ over the interval a to b

Uniform Probability Distribution

- Area under the graph $f(x)$ is equal to 1.
- Area under the graph from 120 to 140

$$= width \times height = 20 \times \frac{1}{20} = 1$$
$$= P(120 \leq x \leq 140)$$

- This property holds for all continuous probability distributions

Normal Probability Distribution

- Most commonly used probability distribution
- Heights and weights of people, test scores, scientific measurements, amounts of rainfall, etc.
- Also used widely statistical inference (sampling)

- Form or shape: bell-shaped distribution
- Symmetric – shape to the left of the mean a mirror image of the shape to the right of mean

Normal Probability Distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where

$\mu = mean$

$\sigma = standard\ deviation$

$\pi = 3.14159$

$e = 2.71828$

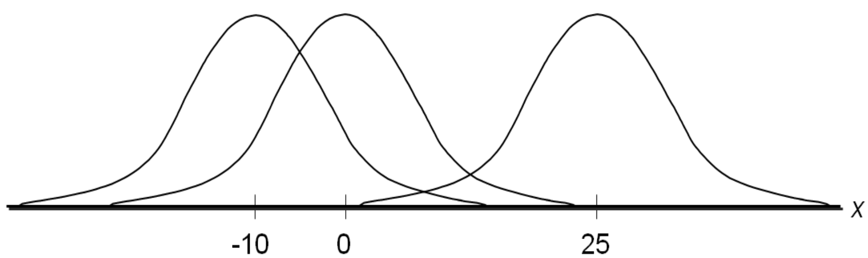
$\pi\ and\ e$: are irrational numbers – each has an infinite number of digits with no pattern or repetition to the right of the decimal point

Normal Probability Distribution

- Two parameters: μ and σ
- The highest point on the normal curve is at the mean, which is also the median and the mode of the distribution
- Mean can be any numerical value. –ve, 0, +ve
- Different distributions may have the same standard deviation with different means, or the same mean with different standard deviations

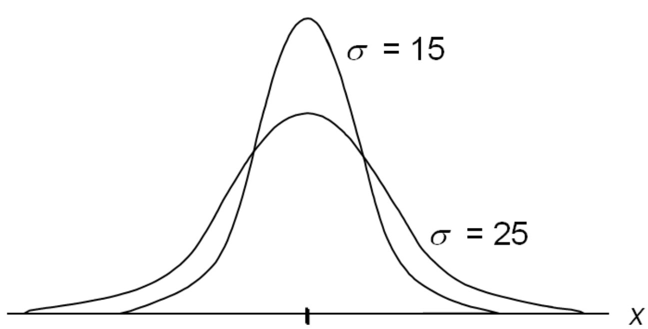
Normal Probability Distribution

- Distributions with the same standard deviation but different means



Normal Probability Distribution

- Distributions with the same mean but different standard deviations



Normal Probability Distribution

- Tails extend to infinity in both directions (theoretically, never touch the horizontal axis)
- Skewness is zero
- Standard deviation σ determines how flat and wide the curve is. Larger the σ, flatter the curve is.
- Area under the curve gives the probability; total area under the curve is 1

Normal Probability Distribution

- Area under the curve gives the probability; total area under the curve is 1
- As the curve is symmetric, the area under the curve to the left of the mean is 0.5 and the area under the curve to the right of the mean is 0.5
- 68.3% of the values within 1 σ of mean
- 95.4% of the values within 2 σ of mean
- 99.7% of the values within 3 σ of mean

Normal Probability Distribution

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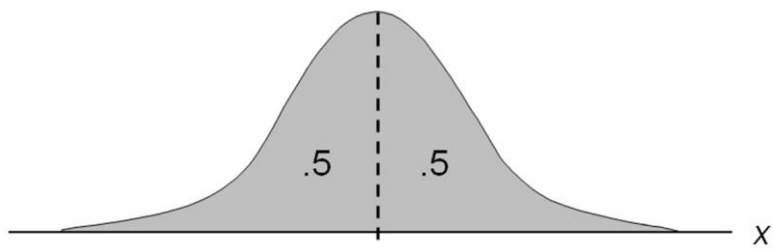

Normal Probability Distribution

- 68.3% of the values within 1 σ of mean
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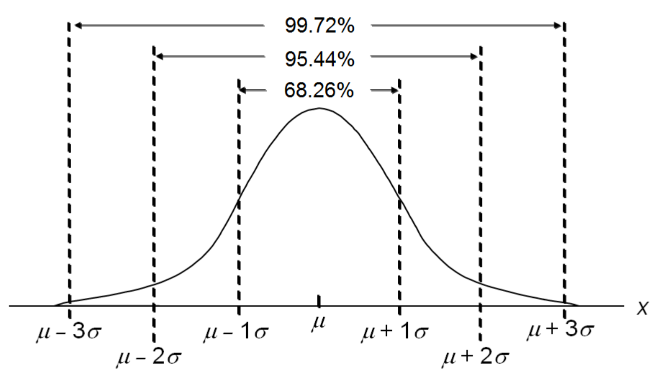
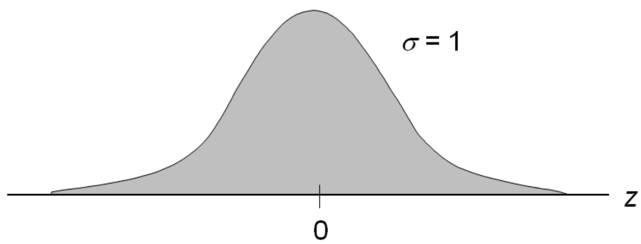


Standard Normal Probability Distribution

- A random variable having a normal distribution with a mean of 0 and a standard deviation of 1 is said to have a standard normal probability distribution.
- The letter z is used to designate the standard normal random variable.



Standard Normal Probability Distribution

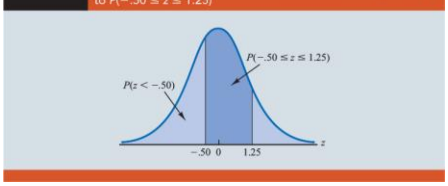
- $P(z \leq 0.83) = 0.7967$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-	-	-	-	-	-	-	-	-	-	-
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.00	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621

Standard Normal Probability Distribution

$P(-0.50 \le z \le 1.25) = P(z \le 1.25) - P(z \le -0.50) = 0.8944 - 0.3085 = 0.5859$

FIGURE 6.9 Cumulative Probability for Normal Distribution Corresponding to $P(-.50 \le z \le 1.25)$

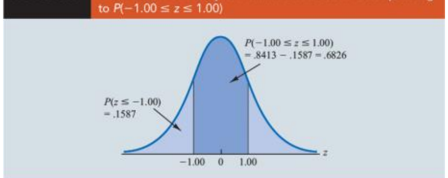


z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177

Standard Normal Probability Distribution

$P(-1.00 \le z \le 1.00) = P(z \le 1.00) - P(z \le -1.00) = 0.8413 - 0.1587 = 0.6826$

FIGURE 6.10 Cumulative Probability for Normal Distribution Corresponding to $P(-1.00 \le z \le 1.00)$

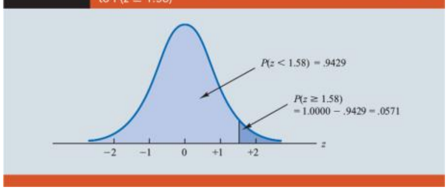


z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441

Standard Normal Probability Distribution

$$P(z \geq 1.58) = 1 - P(z \leq 1.58) = 1 - 0.9429 = 0.0571$$

FIGURE 6.11 Cumulative Probability for Normal Distribution Corresponding to $P(z \geq 1.58)$



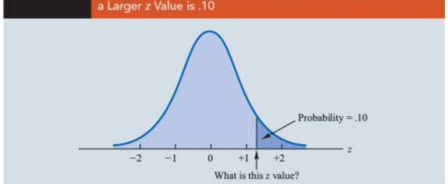
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441

Standard Normal Probability Distribution

Find z_0 such that $P(z \geq z_0) = 0.1$

$$P(z \leq z_0) = 1 - P(z \geq z_0) = 0.9 \Rightarrow z_0 \cong 1.28$$

FIGURE 6.12 Finding z Value such that Probability of Obtaining a Larger z Value is .10



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-	-	-	-	-	-	-	-	-	-	-
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441

$$z_0 = 1.29 - \left[\frac{(0.9015 - 0.9000)}{(0.9015 - 0.8997)} \times (1.29 - 1.28)\right] = 1.282$$

Standard Normal Probability Distribution

- A random variable x that follows a Normal Distribution with mean μ and standard deviation σ can be converted into a standard normal random variable z as

$$z = \frac{x - \mu}{\sigma}$$

- We can think of z as a measure of the number of standard deviations x is from μ.

Standard Normal Probability Distribution

x	z
μ	0
$\mu + \sigma$	$+1$
$\mu + 2\sigma$	$+2$
$\mu - \sigma$	-1
$\mu - 2\sigma$	-2

- We can think of z as a measure of the number of standard deviations x is from μ.

Standard Normal Probability Distribution

- $\mu = 10, \sigma = 2$
- Q. $P(10 \leq x \leq 14) = ?$

x	z
10	0
14	2

$$P(10 \leq x \leq 14) = P(0 \leq z \leq 2)$$
$$= P(z \leq 2) - P(z \leq 0) = 0.9772 - 0.5000$$
$$= 0.4772$$

Grear Tire Company Problem

- Developed a new steel-belted radial tire
- An important sales factor: mileage guarantee
- Looking to devise a “tire mileage guarantee policy”
- Let x = number of miles the tire will last
- From road tests, it is estimated that
- $\mu = 36{,}500, \sigma = 5{,}000$
- A normal distribution is a reasonable assumption

Grear Tire Company Problem

- Q. What % of tires can be expected to last more than 40,000 miles? (or) What is the probability that the tire mileage x will exceed 40,000?
- $x = 40{,}000 \Rightarrow z = \frac{40000-36500}{5000} = 0.70$
- $P(x \geq 40{,}000) = P(z \geq 0.70) = 1 - P(z \leq 0.70) = 1 - 0.7580 = 0.2420$
- 24.2% tires can be expected to last more than 40,000 miles

Grear Tire Company Problem

- Want to provide a discount on replacement tires if the original tire do not provide the guaranteed mileage
- Q. if Grear wants no more than 10% of the tires to be eligible for the discount guarantee, what should be the mileage guarantee?

FIGURE 6.14 Grear's Discount Guarantee

Grear Tire Company Problem

- $P(z \leq -1.28) = 0.10$
- $\therefore x = z\sigma + \mu = (-1.28 \times 5{,}000) + 36{,}500 = 30{,}100$
- Thus, a guarantee of 30,100 miles will meet the requirement that approximately 10% of the tires will be eligible for the guarantee

- The firm may set a mileage guarantee of 30,000 miles (which gives 9.68% eligibility)

Normal Probability Distribution

- Thus, once a probability distribution is established, it can be used to obtain the probability information
- Probability does not make a decision, but provides information to help in decision making

Exponential Probability Distribution

- Random variables – time between arrivals at a hospital emergency room, time required to load a truck, distance between major defects in a highway, and so on.

$$f(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}} \quad for\ x \geq 0$$

where

$\mu = mean$

Exponential Probability Distribution

- $x =$ loading time for a truck at the Schips loading dock and follows such a distribution
- Let $\mu = 15\ minutes$

$$f(x) = \frac{1}{15} e^{-\frac{x}{15}} \quad for\ x \geq 0$$

Q1. $P(x \leq 6)$

Q2. $P(x \leq 18)$

Q3. $P(6 \leq x \leq 18)$

Exponential Probability Distribution

$$P(x \le x_0) = \int_0^{x_0} \frac{1}{\mu} e^{-\frac{x}{\mu}} dx = \left[\frac{1}{\mu} \frac{e^{-\frac{x}{\mu}}}{-\frac{1}{\mu}}\right]_0^{x_0}$$

$$= \left[-e^{-\frac{x}{\mu}}\right]_0^{x_0} = 1 - e^{-\frac{x}{\mu}}$$

$$P(x \le 6) = 1 - e^{-\frac{6}{15}} = 0.3297$$

Exponential Probability Distribution

$$P(x \le x_0) = 1 - e^{-\frac{x}{\mu}}$$

- $P(x \le 6) = 1 - e^{-\frac{6}{15}} = 0.3297$
- $P(x \le 18) = 1 - e^{-\frac{18}{15}} = 0.6988$
- $P(6 \le x \le 18) = 0.6988 - 0.3297 = 0.3691$
- Mean and standard deviation are equal

Poisson vs. Exponential Distribution

Poisson – number of occurrences per interval

$$f(x) = \frac{\mu^x e^{-\mu}}{x!}$$

Exponential – length of interval between occurrences

$$f(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}$$

Poisson vs. Exponential Distribution

Poisson – number of patients during one-hour interval (Let $\mu = 10$)

$$f(x) = \frac{10^x e^{-10}}{x!}$$

Because average number of arrivals is 10 per hour, the average time between arrivals is

$$\frac{1\ hour}{10\ patients} = 0.1\ hour/patient$$

Poisson vs. Exponential Distribution

The corresponding exponential distribution has $\mu = 0.10$

$$f(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}} = \frac{1}{0.1} e^{-\frac{x}{0.1}} = 10e^{-10x}$$

References

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