Continuous Probability Distribution

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Continuous Probability Distributions

- A continuous random variable is a variable that can assume any value on a continuum (can assume an uncountable number of values)
  - ✓thickness of an item
  - ✓time required to complete a task
  - ✓temperature of a solution
  - ✓Height
  - ✓Weight
- These can potentially take on any value, depending only on the ability to measure precisely and accurately.

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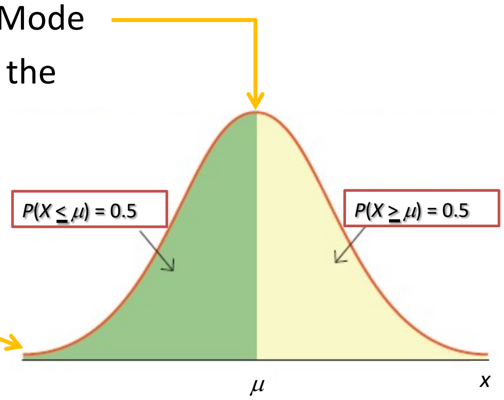
Normal Distribution

- Characteristics of the **Normal Distribution**
  - **Symmetric** about its mean
    - Mean = Median = Mode
  - **Asymptotic**—that is, the tails get closer and closer to the horizontal axis, but never touch it.
  - **Bell-shaped**



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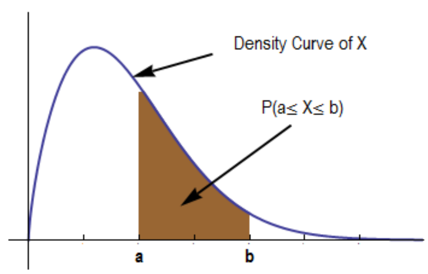
Density Function

- The probability distribution of a **continuous random variable** is represented by a **probability density curve**.
- The probability that *X* has a value in any interval of interest is the area above this interval and below the density curve.



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Continuous Random Variables

- **Probability Density Function** *P*(*x*) of a continuous random variable *X*
  - Describes the relative likelihood that *X* assumes a value within a given interval (e.g., $P(a \leq X \leq b)$), where
    - ***P(x) > 0 for all possible values of X.***
    - ***The area under P(x) over all values of x equals one.***

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Characteristics Normal Distribution

- Characteristics of the **Normal Distribution**
  - The normal probability distribution (Gaussian distribution) is a continuous distribution which is regarded by many as the most significant probability distribution in statistics.
  - The normal distribution is **completely described by two parameters**: μ and σ^2.
    - μ is the population mean which describes the central location of the distribution.
    - σ^2 is the population variance which describes the dispersion of the distribution.

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Normal Distribution Probability Density Function

- The formula for the normal probability density function is

$$P(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{(X-\mu)}{\sigma}\right)^2}$$

Where e = the mathematical constant approximated by 2.71828
π = the mathematical constant approximated by 3.14159
μ = the population mean
σ = the population standard deviation
X = any value of the continuous variable

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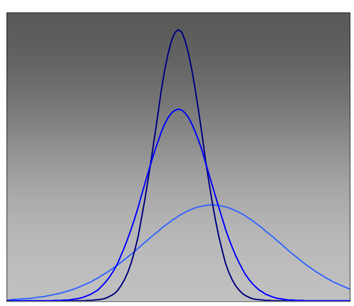
The Normal Distribution



By varying the parameters μ and σ, we obtain different normal distributions

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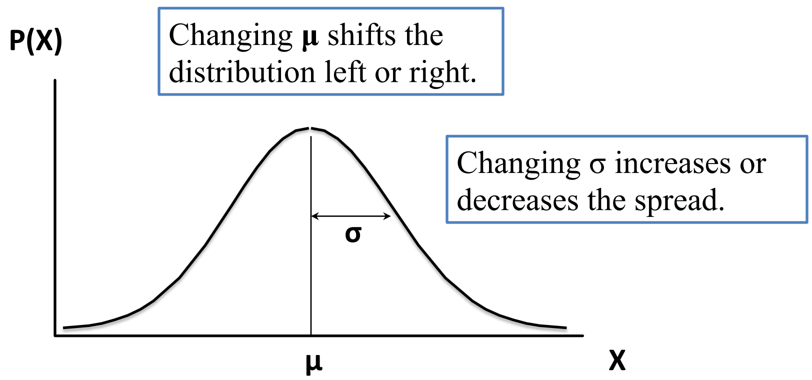
Normal Distribution Shape



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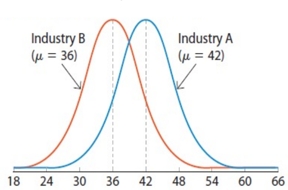
The Normal Distribution

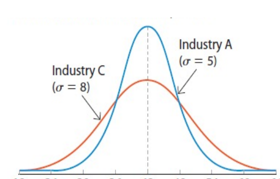
- Example: Suppose the ages of employees in Industries A, B, anad C are normally distributed.
  - Here are the relevant parameters:

Industry A	Industry B	Industry C
$\mu = 42$ years	$\mu = 36$ years	$\mu = 42$ years
$\sigma = 5$ years	$\sigma = 5$ years	$\sigma = 8$ years

  - Let's compare industries using the Normal curves.



σ is the same, μ is different.



μ is the same, σ is different.

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Standardized Normal Distribution

- Any normal distribution (with any mean and standard deviation combination) can be transformed into the **standardized normal distribution (Z).**
- Need to transform X units into Z units.
- The standardized normal distribution has a mean of 0 and a standard deviation of 1.

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Standardized Normal Distribution

- Translate from X to the standardized normal (the "Z" distribution) by subtracting the mean of X and dividing by its standard deviation:

$$Z = \frac{X - \mu}{\sigma}$$

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Standardized Normal Distribution: Density Function

- The formula for the standardized normal probability density function is

$$f(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}}$$

Where e = the mathematical constant approximated by 2.71828

π = the mathematical constant approximated by 3.14159

Z = any value of the standardized normal distribution

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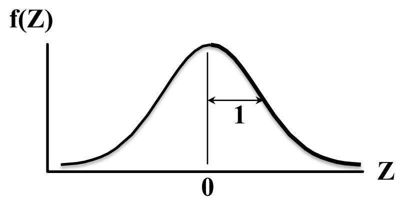
Standardized Normal Distribution: Shape

- Also known as the "Z" distribution
- Mean is 0
- Standard Deviation is 1



Values above the mean have positive Z-values, values below the mean have negative Z-values

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Standardized Normal Distribution: Example

- If X is distributed normally with mean of 100 and standard deviation of 50, the Z value for X = 200 is

$$Z = \frac{X - \mu}{\sigma} = \frac{200 - 100}{50} = 2.0$$

- This says that X = 200 is two standard deviations (2 increments of 50 units) above the mean of 100.

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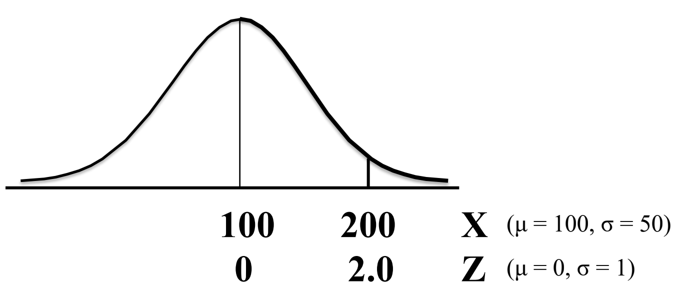
Standardized Normal Distribution: Example



Note that the distribution is the same, only the scale has changed. We can express the problem in original units (X) or in standardized units (Z)

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The Standard Normal Distribution

- **Standard Normal Table (*Z* Table)**.
  - Gives the cumulative probabilities $P(Z \leq z)$ for positive and negative values of z.
  - Since the random variable Z is symmetric about its mean of 0,

$$P(Z < 0) = P(Z > 0) = 0.5.$$

  - To obtain the $P(Z < z)$, read down the z column first, then across the top.

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The Standard Normal Distribution

- Standard Normal Table (*Z* Table).

Table for positive *z* values.

z	0.00	0.01	0.02	0.03	0.04
0.0	0.5000	0.5040	0.5080	0.5120	0.5160
0.1	0.5398	0.5438	0.5478	0.5517	0.5557
0.2	0.5793	0.5832	0.5871	0.5910	0.5948
0.3	0.6179	0.6217	0.6255	0.6293	0.6331
0.4	0.6554	0.6591	0.6628	0.6664	0.6700

Table for negative *z* values.

z	0.00	0.01	0.02	0.03	0.04
-3.9	0.0000	0.0000	0.0000	0.0000	0.0000
-3.8	0.0001	0.0001	0.0001	0.0001	0.0001
-3.7	0.0001	0.0001	0.0001	0.0001	0.0001
-3.6	0.0002	0.0002	0.0001	0.0001	0.0001
-3.5	0.0002	0.0002	0.0002	0.0002	0.0002

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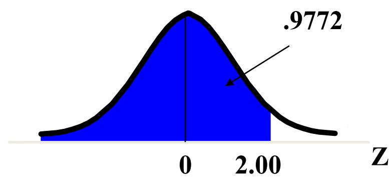
Normal Probability Tables

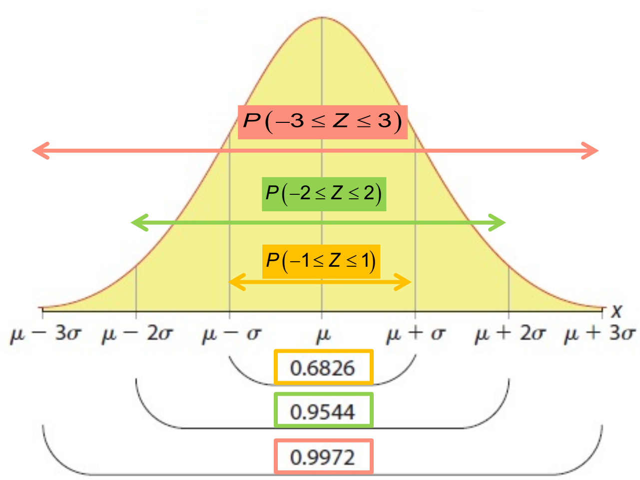
- The Standardized Normal table in the textbook gives the probability less than a desired value for Z (i.e., from negative infinity to Z)

Example:
P(Z < 2.00) = .9772



The Standard Normal Distribution

- Revisiting the **Empirical Rule**.



The Standard Normal Distribution

- Example: The **Empirical Rule**
  - An investment strategy has an expected return of 4% and a standard deviation of 6%. Assume that investment returns are normally distributed.
  - What is the probability of earning a return greater than 10%?
    - A return of 10% is one standard deviation above the mean, or 10 = $\mu + 1\sigma$ = 4 + 6.
    - Since about 68% of observations fall within one standard deviation of the mean, 32% (100% – 68%) are outside the range.

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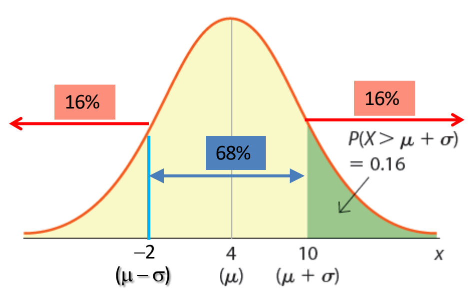
The Standard Normal Distribution

- Example: The **Empirical Rule**
  - An investment strategy has an expected return of 4% and a standard deviation of 6%. Assume that investment returns are normally distributed.
  - What is the probability of earning a return greater than 10%?
    - Using symmetry, we conclude that 16% (half of 32%) of the observations are greater than 10%.



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Demand for Salmon

- Akiko Hamaguchi, manager of a small sushi restaurant, Little Ginza, in Phoenix, Arizona, has to estimate the daily amount of salmon needed.
- Akiko has estimated the daily consumption of salmon to be normally distributed with a mean of 12 pounds and a standard deviation of 3.2 pounds.
- Buying 20 lbs of salmon every day has resulted in too much wastage.
- Therefore, Akiko will buy salmon that meets the daily demand of customers on 90% of the days.

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Demand for Salmon

- Based on this information, Akiko would like to:
  - Calculate the proportion of days that demand for salmon at Little Ginza was above her earlier purchase of 20 pounds.
  - Calculate the proportion of days that demand for salmon at Little Ginza was below 15 pounds.
  - Determine the amount of salmon that should be bought daily so that it meets demand on 90% of the days.

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Solving Problems with the Normal Distribution

- We can now answer the questions first posed by Akiko Hamaguchi in the introductory case of this chapter. Recall that Akiko is concerned about buying the right amount of salmon for daily consumption at Little Ginza. Akiko has estimated that the daily consumption of salmon is normally distributed with a mean of 12 pounds and a standard deviation of 3.2 pounds. She wants to answer the following questions:
  - **a.** What proportion of days was the demand at Little Ginza above her earlier purchase of 20 pounds?

SOLUTION: Let X denote consumer demand for salmon at the restaurant. We know that X is normally distributed with $\mu = 12$ and $\sigma = 3.2$.

a. $P(X > 20) = P\left(Z > \frac{20 - 12}{3.2}\right) = P(Z > 2.50) = 1 - 0.9938 = 0.0062.$

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Solving Problems with the Normal Distribution

- We can now answer the questions first posed by Akiko Hamaguchi in the introductory case of this chapter. Recall that Akiko is concerned about buying the right amount of salmon for daily consumption at Little Ginza. Akiko has estimated that the daily consumption of salmon is normally distributed with a mean of 12 pounds and a standard deviation of 3.2 pounds. She wants to answer the following questions:
  - **a.** What proportion of days was the demand at Little Ginza above her earlier purchase of 20 pounds?

  - **b.** What proportion of days was the demand at Little Ginza below 15 pounds?

b. $P(X < 15) = P\left(Z < \frac{15 - 12}{3.2}\right) = P(Z < 0.94) = 0.8264.$

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Solving Problems with the Normal Distribution

- We can now answer the questions first posed by Akiko Hamaguchi in the introductory case of this chapter. Recall that Akiko is concerned about buying the right amount of salmon for daily consumption at Little Ginza. Akiko has estimated that the daily consumption of salmon is normally distributed with a mean of 12 pounds and a standard deviation of 3.2 pounds. She wants to answer the following questions:

  c. How much salmon should she buy so that it meets customer demand on 90% of the days?

  c. In order to compute the required amount of salmon, we solve for x in $P(X \leq x) = 0.90$. Since $P(X \leq x) = 0.90$ is equivalent to $P(Z \leq z) = 0.90$, we first derive $z = 1.28$. Given $x = \mu + z\sigma$, we find $x = 12 + 1.28(3.2) = 16.10$. Therefore, Akiko should buy 16.10 pounds of salmon daily to ensure that customer demand is met on 90% of the days.

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Assessing Normality

- It is important to evaluate how well the data set is approximated by a normal distribution.
- Normally distributed data should approximate the theoretical normal distribution:
  - The normal distribution is bell shaped (symmetrical) where the mean is equal to the median.
  - The empirical rule applies to the normal distribution.

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Assessing Normality

- Construct charts or graphs
  - For small- or moderate-sized data sets, box-and-whisker plot look symmetric?
  - For large data sets, does the histogram or polygon appear bell-shaped?
- Evaluate normal probability plot (**Q-Q Plot)**
  - Is the normal probability plot approximately linear with positive slope?

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The Normal Probability Plot

- Normal probability plot (steps):
  - ✓Arrange data into ordered array
  - ✓Find corresponding standardized normal quantile (Z) values
  - ✓Plot the pairs of points with observed data values (X) on the vertical axis and the standardized normal quantile (Z) values on the horizontal axis
  - ✓Evaluate the plot for evidence of linearity

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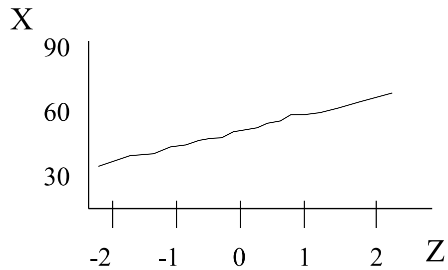
The Normal Probability Plot

A normal probability plot for data from a normal distribution will be approximately linear:



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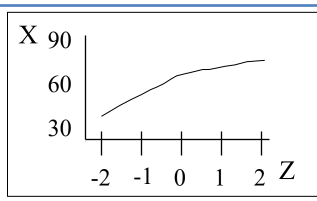
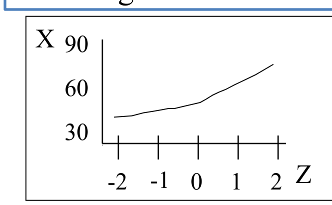
The Normal Probability Plot

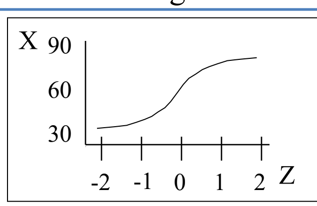
Left-Skewed



Right-Skewed



Rectangular



Non-linear plots indicate a deviation from normality

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THANK YOU

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