CORE CURRICULUM

Finance
Mihir A. Desai, Series Editor

READING + INTERACTIVE ILLUSTRATIONS

Financial Options

and Their Application to Corporate Finance

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1 INTRODUCTION

This reading covers the basic elements of financial options and introduces the analytical techniques used to determine their value. A financial option is a derivative security whose value is dependent on (or derives from) the value of an underlying asset. The underlying asset may be a commodity, a stock, a currency, an interest rate, or some combination of assets, such as a stock index. For example, a derivative in soybeans derives its value from the price of soybeans, while a stock derivative derives its value from the price of a stock.

A derivative security need not be a mysterious financial instrument. Financial derivatives are simply contracts written between a buyer and a seller to exchange an asset at a future date at an agreed price. Farmers and merchants around the world have long used derivatives to insure against fluctuations in the price of their crops, and economic historians have identified derivative contracts in the laws of ancient Mesopotamia. There are many different types of financial derivatives, including forward contracts, swaps, futures, and options. This reading focuses on financial options because of their importance to corporate finance.

An *option* is the right, but not the obligation, to buy or sell an underlying asset at a future date at a specific price. Businesses, consumers, and investors routinely use options, either explicitly or implicitly; for example, an airline may use options to place an upper limit on the price it pays for jet fuel; a family contemplating a new home purchase may "lock in" the mortgage interest rate during the time it takes to complete the transaction; and an investor may secure a capital gain on a particular stock without liquidating the position. In this reading, we show how options are used to manage risk and to implement specific financial strategies. The reading focuses on understanding the basics of financial options and determining their value.

The material in the reading is presented in two parts. The core concepts for understanding financial options are presented in the Essential Reading section, and extensions and more advanced concepts are presented in the Supplemental Reading section.

In the Essential Reading section, we first develop a foundation for understanding the fundamentals of financial options, including the basic terminology used to describe option contracts and the techniques used to understand the cash flow implications of entering into an option contract. Using this foundation, we develop a framework for valuing options based on the idea that, in a perfect market, the value of an option will equal the cost of constructing a portfolio of assets that reproduces (or replicates) the option's future cash flows. We then extend the framework to a dynamic setting as the basis for the widely adopted Black-Scholes-Merton (or simply "Black-Scholes") formula, which is based on the work of Fisher Black, Myron Scholes, and Robert Merton. We show how to use the Black-Scholes model to calculate the value of a financial option.

In the Supplemental Reading section, we demonstrate the use of the Black-Scholes formula in a variety of corporate finance applications (the valuation of an employee stock option grant, the evaluation of the terms in an acquisition offer, and the determination of an appropriate yield on a new debt issue) and discuss extensions of the framework developed in the Essential Reading section.

2 ESSENTIAL READING

2.1 Fundamental Concepts

2.1.1 Basics of Financial Options

There are two basic types of financial options, **call options** and ***put options***. A call option gives the ***option buyer*** (or the ***option holder***) the right—but not the obligation—to purchase the underlying asset at a specific price at or before a particular date in the future (the ***expiration date***). A put option gives the buyer the right—but not the obligation—to sell the underlying asset at a specific price at or before a particular date in the future. The option buyer pays a fee, called the ***premium***, to the ***option seller*** (or the ***option writer***) for the right to buy, or sell, the underlying asset. The contracted price at which the underlying asset may be purchased or sold in the future is called the ***strike price*** or the ***exercise price***. Because a financial option is a contract written between two parties, the payments and outcomes are zero-sum; the buyer pays the premium to the seller, and over the duration of the contract, what one party gains, the other party loses, and vice-versa. The options contract simply specifies an allocation of payments and future outcomes between the two parties.

Options are traded both on organized exchanges, such as the Chicago Board of Options Exchange (CBOE), and in customized, over-the-counter (OTC) transactions. The option contract specifies the number of shares, the price, and the duration of the contract. If the holder of the option chooses to act on the right to buy or sell the underlying shares, the option holder is said to ***exercise*** the option. Option contracts that can be exercised anytime during the contract period are referred to as ***American options***. Option contracts that can be exercised only at the end of the contract period (or at expiration) are referred to as ***European options***. (These terms were coined by Paul Samuelson and refer to complexity, not geography.[a]) Because European options are conceptually simpler than American options, we will focus our analysis primarily on constructing a framework for understanding European options. Our discussion of European options will identify and develop the key fundamentals of options. In the Supplemental Reading section, we extend the framework to understand the differences between American options and European options—and more importantly, when those differences may or may not be important.

Option Buyers

The buyer or holder of an option is said to have a ***long position*** in the contract. Because an option contract conveys the right—but not the obligation—to exercise the option, the option holder will exercise the option only when it is beneficial to do so. A call option will never be exercised when the price of the underlying asset is less than the strike price because the holder can instead choose to purchase the asset in the marketplace at a lower price. Similarly, a put option will never be exercised when the price of the underlying asset is greater than the strike price because the holder can instead choose to sell the asset in the marketplace at a higher price. A key feature of an options contract is that the holder is never obligated to exercise the option.

Because the holder of an option is not obligated to exercise the option, the decision to exercise (or not) is based on the price of the underlying asset relative to the specified strike price. A call option is said to be ***out-of-the-money*** when the price of the underlying asset is *less than* the strike price and ***in-the-money*** when the price of the underlying asset is *greater than* the strike price. Conversely, a put option is out-of-the-money when the price of the underlying asset is *greater than* the strike price and in-the-money when the price of the underlying asset is *less than* the strike price. When the price of the underlying asset is equal to the strike price, both call options and put options are said to be ***at-the-money***. Out-of-the money options are never rationally exercised.

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[a] See Robert C. Merton, "Paul Samuelson and Financial Economics," *American Economist* 81, no. 1 (2006): 19, 28.

Option Sellers

The seller or writer of an option is said to have a ***short position*** in the contract. The option writer receives the premium paid by the buyer and assumes the obligation to fulfill the terms of the contract should the option holder exercise the option. The premium compensates the option writer for accepting the risks associated with writing the option contract. The writer of a call option takes on the risk that the option ends up *in-the-money*, such that the price of the underlying asset is higher than the strike price. In this case, the call option will be exercised and the option writer will be obligated to deliver the underlying asset at the lower strike price. Since there is no limit on how much the price of the underlying asset might increase, the writer of a call option—who does not also own the underlying asset—potentially faces unlimited losses. The writer of a put option takes on the risk that the option ends up *in-the-money*, such that the price of the underlying asset is lower than the strike price. In this case, the put option will be exercised and the option writer will be obligated to purchase the underlying asset from the seller at the higher strike price. In contrast to a call option, the potential loss to the writer of a put contract is limited and is never more than the specified strike price.

Exhibit 1 summarizes when an option is considered to be in-the-money, at-the-money, or out-of-the-money. It also indicates the actions taken by the buyers and sellers of European calls and puts at expiration.

EXHIBIT 1 Actions Taken by Buyers and Sellers of European Calls and Puts at the Expiration Date

Call Option Contract				Put Option Contract		
Seller (writer)	**Buyer** (holder)				**Seller** (writer)	**Buyer** (holder)
Obligated to sell stock to Buyer	Will choose to exercise option. Buys stock from Seller and then sells on stock market at a higher price.	In the money	STOCK PRICE **ABOVE** EXERCISE PRICE	Out of the money	No action needed	Will not exercise option
		At the money	STOCK PRICE **EQUAL TO** EXERCISE PRICE	At the money		
No action needed	Will not exercise option	Out of the money	STOCK PRICE **BELOW** EXERCISE PRICE	In the money	Obligated to buy stock from Buyer	Will choose to exercise option. Buys stock on stock market and then sells to the Seller for a higher price.

It is useful to consider options from the perspectives of both the buyer and seller because it helps to explain how markets in financial options work. Buyers of financial options pay a premium to guard against or benefit from changes in the market values of an underlying asset. Sellers of financial options collect premiums that compensate for the risk of making payouts when (and if) the options are exercised. For a market in financial options to function, option premiums must be attractive to both option buyers and option sellers.

2.1.2 Payoffs to Option Contracts

An option contract specifies a set of *right*s (for option holders) and *obligations* (for option writers) over the life of the contract. The specific outcome (or *payoff*) from entering into an option contract will be determined by the realized future price of the underlying asset. Because the future price of the asset is not known, it is important to understand how the option's payoff will relate to the realized price of the underlying asset. ***Payoff functions*** and ***payoff diagrams*** capture this relationship. It is important to note that the potential payoff from a particular option contract is different from the profit that is realized from entering into the contract. The realized profit from buying an option contract is equal to the contract's eventual payoff less the premium paid to purchase the contract. Similarly, the realized profit from selling an option contract is equal to the premium received when the contract is written less the eventual payout (if any) made during the life of the contract.

Payoff Functions and Diagrams

The following section shows how payoff functions and diagrams work and provides an opportunity to test your understanding of the key terms introduced in the previous section. To demonstrate the construction and use of payoff functions and diagrams, we will analyze option contracts written on the stock of CleanPower Energy, Inc., a publicly traded corporation dedicated to the global development of sustainable, clean energy projects.[b]

Call Option Payoffs

Assume you buy a *European* call option on CleanPower stock with a $25 *strike price*. In other words, you have a *long position* in the call option (or you are "*long* the call option"). If CleanPower's stock price is $50 on the *expiration date*, your option will be *in-the-money* by $25. Because the *option writer* is obligated to sell you a share of CleanPower stock at the $25 *strike price*, you can *exercise* the option, purchase the share of CleanPower for $25 from the *option writer*, and then sell the purchased

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[b] CleanPower Energy, Inc., is a fictitious company, as are the other companies in this reading—Simple Solutions, Dynamic Enterprises, Continuous Media Corporation and EmpireCo—examples used to illustrate concepts.

CleanPower share for \$50 on the stock exchange. After this sequence of transactions, your payoff would equal \$25 = \$50 − \$25.

However, if CleanPower's stock price is only \$15 on the *expiration date*, your option will be *out-of-the-money* by \$10. The option gives you the *right* to purchase a share of CleanPower stock from the *option writer* (who is *short* the call), but you are *not obligated* to do so. In fact, it would not make any sense to purchase the CleanPower share from the *option writer* for \$25 when you could purchase one for \$15 from the stock exchange. In this case, you would *not exercise* your option and would instead let the option expire. Your *payoff* would equal zero.

Based on the previous example, we can use the following formula, called a *payoff function*, to summarize the payoffs to the holder of a call option:

Call Holder's Payoff Function

$$C_T = \max\left(S_T - X, 0\right)$$

where C_T is the payoff to the holder of the call option on the contract's expiration date T, S_T is the price of the underlying asset on the expiration date, and X is the option's strike price. Because the outcomes for the two contracting parties are zero-sum, the payoff for the option *writer* of a call option is summarized by the formula

Call Writer's Payoff Function

$$-C_T = -\max\left(S_T - X, 0\right) = \min\left(X - S_T, 0\right)$$

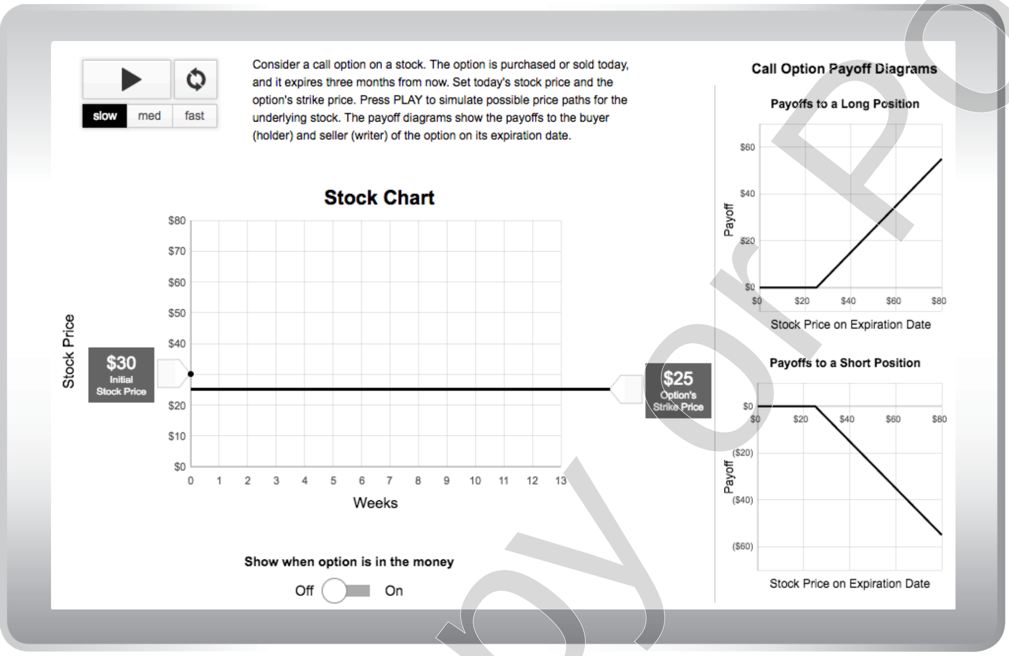
Interactive Illustration 1 illustrates *payoff diagrams* for call options. To see payoffs for the holder and writer of the CleanPower call option, keep the strike price at the default value of \$25. The graph at the upper right shows the payoff diagram for the long position in the call option (the holder's payoff), and the graph at the lower right shows the payoff diagram for the short position in the call option (the writer's payoff).

Note that the call option is only exercised if it is in-the-money (stock price > strike price) on the expiration date. For every dollar over the strike price, the long position earns a dollar and the short position loses a dollar. Run the simulated stock price paths to see how the value of a call option is non-zero only if the stock price is higher than the strike price.

The Stock Chart in Interactive Illustration 1 simulates the price of the underlying asset *over time*, ending at the call option contract's expiration date. On the other hand, the payoff diagrams (at the right) show the payoffs to the long and short positions at a *single point in time*, that is, on the expiration date.

INTERACTIVE ILLUSTRATION 1 Call Option Payoff Diagrams

Scan this QR code, click the image, or use this link to access the interactive illustration: bit.ly/hbsp2pIap7b



Consider a call option on a stock. The option is purchased or sold today, and it expires three months from now. Set today's stock price and the option's strike price. Press PLAY to simulate possible price paths for the underlying stock. The payoff diagrams show the payoffs to the buyer (holder) and seller (writer) of the option on its expiration date.

Put Option Payoffs

Similar to the analysis of call option payoffs, the following example shows how the payoff function for a put option works. Assume you buy a *European* put option on CleanPower stock with a $35 *strike price*. If CleanPower's stock price is $15 on the expiration date, your option will be *in-the-money* by $20. Because the *option writer* (who has a short position in the option) is obligated to buy your share of CleanPower stock at the $35 strike price, you can buy a share of CleanPower on the stock exchange for $15, exercise the option, and sell the purchased CleanPower share to the option writer for $35. After this sequence of transactions, your payoff would equal $20 = $35 – $15.

However, if CleanPower's stock price is $50 on the *expiration date*, your option will be *out-of-the-money* by $15. The option gives you the *right* to sell a share of CleanPower stock to the *option writer*, but you are *not obligated* to do so. In fact, it would not make any sense to purchase a CleanPower share for $50 from the stock exchange in order to sell it to the *option writer* for $35. In this case, you would *not exercise* your option and would, instead, let the option expire. Your *payoff* would equal zero.

Based on the previous example, we can write the payoff function that summarizes the payoffs to the *holder* of a put option:

Put Holder's Payoff Function

$$P_T = \max\left(X - S_T, 0\right)$$

where P_T is the payoff to the holder of the put option on the expiration date and the other variables are defined as in the previous example. Because the outcomes for the two contracting parties are zero-sum, the payoff for the *option writer* of a put option is summarized by the formula

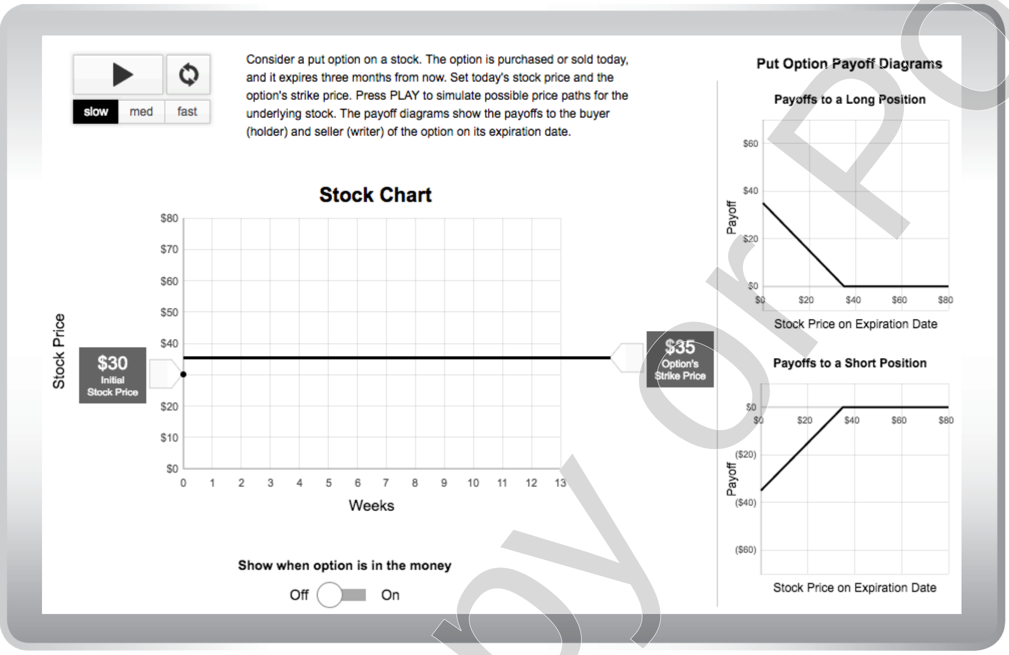
Put Writer's Payoff Function

$$-P_T = -\max\left(X - S_T, 0\right) = \min\left(S_T - X, 0\right)$$

Interactive Illustration 2 shows payoff diagrams for put options on the right side of the screen. To see payoffs for the holder and writer of the CleanPower put option, keep the strike price at the default $35. The upper graph shows the payoff diagram for the long position in the put option, and the lower graph shows the payoff diagram for the short position in the put option. Note that the put option is only exercised if it is in-the-money (stock price < strike price) on the expiration date. For every dollar under the strike price, the long position earns a dollar and the short position loses a dollar, but gains for the holder and losses for the writer are limited. The stock price can't go lower than zero. Run the simulated stock price paths to see how the value of a put option is non-zero only if the stock price is lower than the strike price.

INTERACTIVE ILLUSTRATION 2 Put Option Payoff Diagrams

Scan this QR code, click the image, or use this link to access the interactive illustration: bit.ly/hbsp2DYYdDg


Consider a put option on a stock. The option is purchased or sold today, and it expires three months from now. Set today's stock price and the option's strike price. Press PLAY to simulate possible price paths for the underlying stock. The payoff diagrams show the payoffs to the buyer (holder) and seller (writer) of the option on its expiration date.

To summarize, payoff formulas for holders and writers of call and put options are

	Long Position (Holder)	**Short Position** (Writer)
Call	$C = \max(S - X, 0)$	$-C = \min(X - S, 0)$
Put	$P = \max(X - S, 0)$	$-P = \min(S - X, 0)$

Stock and Bond Payoffs

Because an option provides the right to buy or sell the underlying stock at a future date, it is important to also consider the potential future payoffs to holding the *underlying stock* itself over the life of the option. Because a European option's payoff occurs at the expiration date, we will study the payoffs to holding the underlying stock until the expiration of the option contract. We will also assume that the underlying stock does not pay any dividends over this period, such that the stock price at the option expiration date represents the entire payoff to holding the stock.

Because an option contract relates to a potential transaction in the future, an investor must also consider the rate of interest that could be earned on a risk-free investment over the life of the option. For this reason, it is important to understand the payoffs to a risk-free bond that matures at the expiration date of the option contract. A European option's payoff occurs at the expiration date, so we will consider the payoffs to a ***zero-coupon bond***, where the entire payoff to holding the bond occurs at the option expiration date. We will use payoff diagrams to summarize the payoffs associated with holding a stock or a risk-free, zero-coupon bond over the life of an option contract.

Assume you own one share of CleanPower stock (you have a *long position*). Because the future price of CleanPower stock is not known, you face a range of potential outcomes. Since you have not paid for the stock, your payoff from a long position in CleanPower stock is equal to the future stock price.

Now think about the payoffs to a *short position* in CleanPower stock. Assume that you are short one share of CleanPower stock, which means that you owe someone one share of CleanPower stock. Your payoff diagram is negative because you no longer have the CleanPower share and must purchase a replacement share to repay the debt. If the price of CleanPower is $10, you lose $10; if the price is $50, you lose $50.

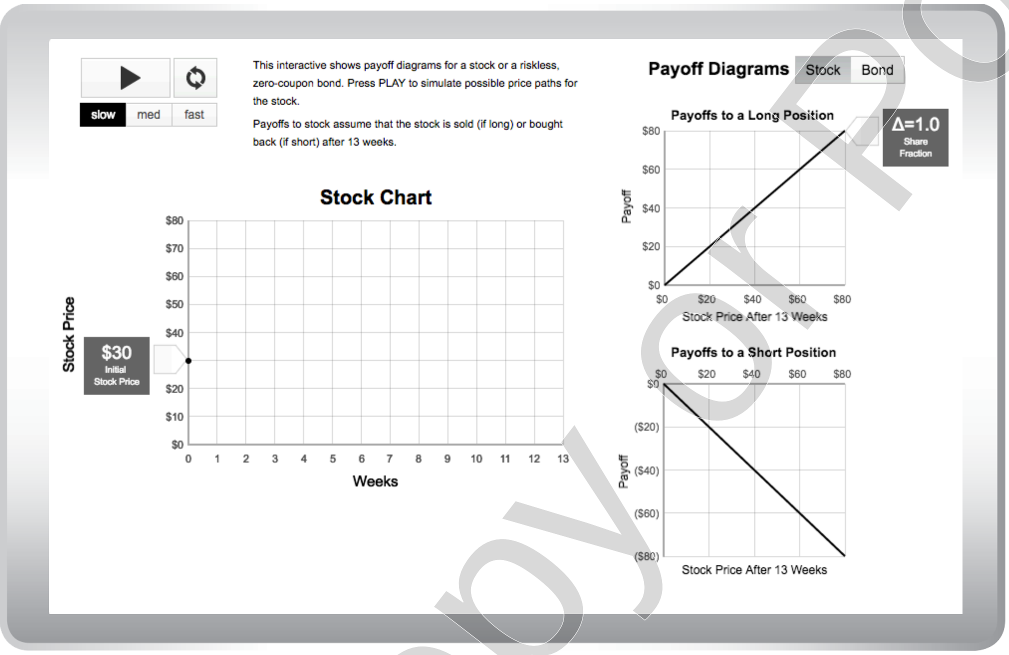
Just as you can construct a payoff diagram for a position in a stock, you can also construct a payoff diagram for a position in a bond. For example, assume you buy a zero-coupon bond (you have a *long position*) with a face value of $25 and assume the bond has no risk, meaning that you are guaranteed to receive $25 when the bond matures. Because the bond's payout is guaranteed, its payout function is independent of the CleanPower stock price, and the bond's payoff to a long position is constant at $25. Similarly, the seller of the bond (who has a *short position*) is guaranteed to pay $25 when the bond matures. Because the bond's payout is guaranteed, the payoff to a short position is constant at −$25.

Like Interactive Illustrations 1 and 2, **Interactive Illustration 3** shows payoff diagrams at the right side of the screen. You can toggle (at the upper right) between stock payoff diagrams and bond payoff diagrams. Using the stock price animation, convince yourself that the payoff diagram for a long position in stock starts at 0 and has a slope of 1, and the payoff diagram for a short position in stock starts at 0 and has a slope of negative 1. The payoff diagram for a bond is equal to the bond's face value and is positive for a long position and negative for a short position.

INTERACTIVE ILLUSTRATION 3

Payoff Diagrams for a Stock and a Riskless Bond

Scan this QR code, click the image, or use this link to access the interactive illustration: bit.ly/hbsp2pHwimX



This interactive shows payoff diagrams for a stock or a riskless, zero-coupon bond. Press PLAY to simulate possible price paths for the stock.

Payoffs to stock assume that the stock is sold (if long) or bought back (if short) after 13 weeks.

Option Prices

The price of an option contract derives from its potential future payoffs. An option's current value equals the present discounted value of the cash flows (or payoffs) the contract is expected to provide in the future. In a competitive market, an option's price will equal its current value. If the price of an option were less than its current value, we would expect its price to increase as investors bid to purchase "cheap" future cash flows. As a result, competition would drive the price of the option contract up to the point at which its price was equal to the present discounted value of the option's expected future payoff. Similarly, if the price of an option were greater than its current value, we would expect its price to decrease as investors attempted to sell the "expensive" contract. As a result, competition would drive the price of the option contract down to the point at which its price was equal to the present discounted value of the option's expected future payoff.

Based on the above logic, in a competitive market, the price of an option contract will equal the value of an option contract. We will use the letter C to represent the price (or value) of a call option and the letter P to represent the price (or value) of a put option. We will also distinguish between *option value* and ***intrinsic value***. Option value equals the present discounted value of the option's expected future payoffs given the current stock price—in other words, the option's current value or price.

Intrinsic value is the hypothetical value of the option "as if" the option were currently at expiration—in other words, the option's payoff at the current stock price. Because an option's value is determined by its potential future payoffs, it is always non-negative—even if the option is currently out-of-the-money and the intrinsic value of the option is zero.

Given the price of an option contract, we can refer to the *profit* associated with buying or selling an option contract as

$$\text{Profit} = \text{Payoff} - \text{Price}$$

In similar fashion, we can refer to the return associated with buying or selling an option contract as

$$\text{Return} = \frac{\text{Profit}}{\text{Price}} = \frac{\text{Payoff} - \text{Price}}{\text{Price}}$$

Creating Structured Payoffs

Option contracts can be combined, together with a position in the underlying stock and/or a position in a riskless bond, to create portfolios with structured payoffs. For example, an investor who owns stock in a pharmaceutical company might want to protect (or *hedge*) against the possibility that the value of the long position in stock could fall significantly—perhaps the company's stock price had risen significantly from \$20 to \$35 on speculation that the company was about to announce a new blockbuster drug, and the investor might be concerned that management could issue a statement disproving the speculation. To hedge his position in the stock and limit his downside risk, the investor might purchase a put option with a strike price of \$30. In this case, the investor would now hold a portfolio of one share of stock and one put contract. His portfolio would have the following components and associated payoffs:

Portfolio Components

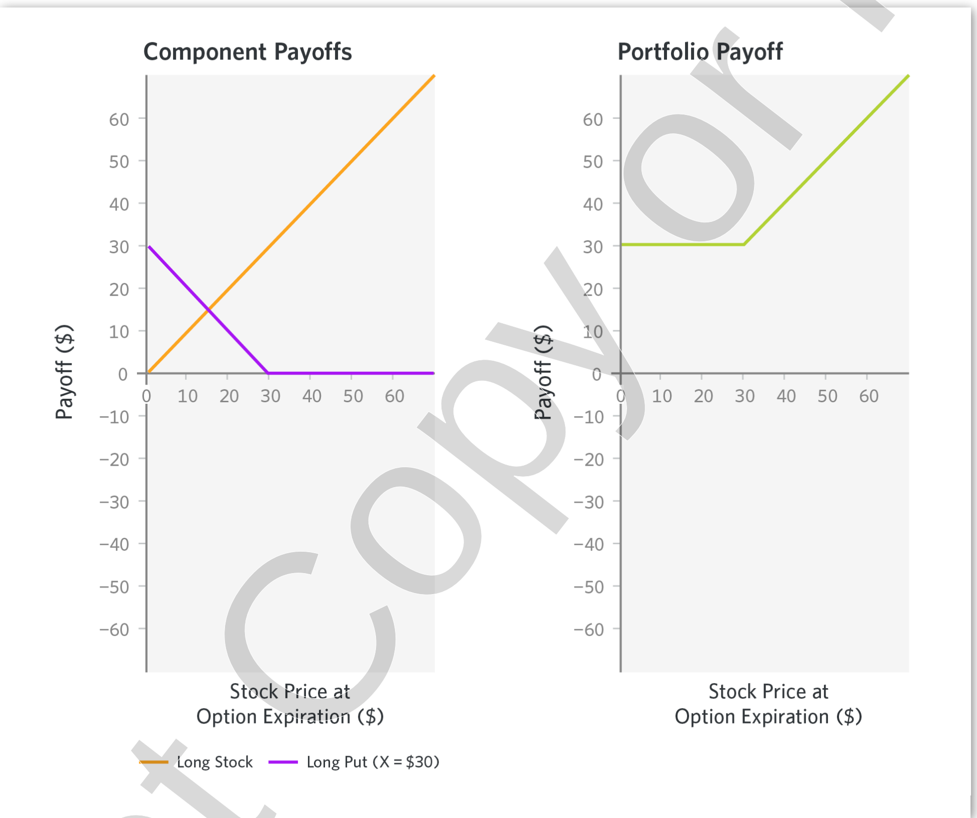
$$\text{Long Stock} + \text{Long Put (with Strike} = \$30)$$

Portfolio Payoffs

$$\text{Portfolio}_T = S_T + \max(\$30 - S_T, 0) = \begin{cases} \$30 & \text{if } S_T \leq \$30 \\ S_T & \text{if } S_T > \$30 \end{cases}$$

Exhibit 2 shows payoffs for the individual components and the combined portfolio. The left-hand panel shows the payoff diagrams for one share of stock and one put contract, and the right-hand panel shows the payoff diagram for the combined portfolio. You can see that between $0 and $30 on the horizontal axis, the payoffs of the stock and put sum to $30.

EXHIBIT 2 Creating a Portfolio with Limited Downside Risk



By purchasing the put option, the investor has created a combined portfolio with a guaranteed "floor" value of $30 per share. However, this hedged position has come at a price. The investor must pay $P to purchase the put contract. In other words, the investor used a put contract as a form of insurance against a fall in the value of his stock holding. The price of the put contract is equivalent to the premium on an insurance policy.[c]

Consider another example: A company founder might want to limit the risk of holding a concentrated position in his company's stock. Outside investors can lower

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[c] Note that the premium is not included in Exhibit 2 because the figure shows the payoffs to the contract, not the profit from entering into the contract.

the risk of holding a particular stock by diversifying, but such diversification may be difficult for a corporate insider. If the company founder were to sell a substantial amount of his holdings, investors may interpret this as a negative signal about the firm's prospects, resulting in a decline in the company's valuation. To limit his exposure to fluctuations in the price of his company's stock without liquidating his position, the founder could construct a portfolio with the following components and payoffs:

Portfolio Components

$$\text{Long Stock} + \text{Long Put } (\text{with Strike} = \$30) + \text{Short Call } (\text{with Strike} = \$60)$$

Portfolio Payoffs

$$\text{Portfolio}_T = S_T + \max(\$30 - S_T, 0) + \min(\$60 - S_T, 0) = \begin{cases} \$30 & \text{if } S_T \leq \$30 \\ S_T & \text{if } \$30 < S_T < \$60 \\ \$60 & \text{if } S_T \geq \$60 \end{cases}$$

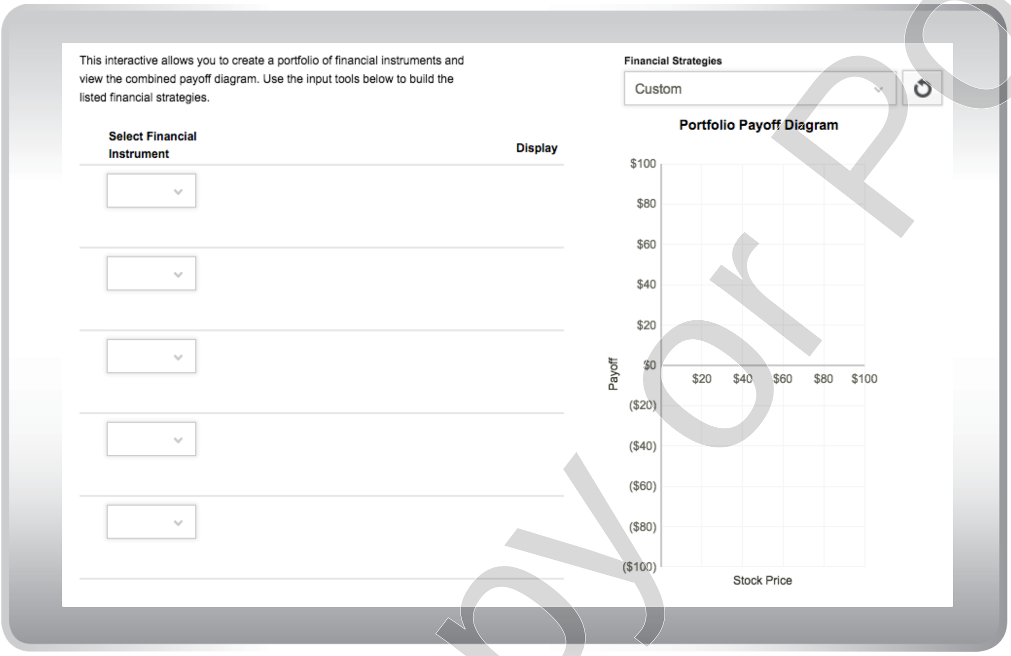
By selling a call option with a strike price of $60 and using the proceeds to purchase a put contract with a strike price of $30, the company founder can create a combined portfolio with a value that is guaranteed to be between $30 and $60. In addition, by adjusting the strike prices of the put and call contracts, the founder can arrive at a net cost of constructing the hedge of $P − $C (the cost of purchasing the put option in excess of the money received from selling the call option) that is less than the estimated price impact of liquidating his position in the stock. This strategy, commonly referred to as a *collar*, creates a hedge against volatility.

Interactive Illustration 4 allows you to create all kinds of structured payoffs. At the upper right of the interactive, select "Collar" from the list of Financial Strategies. The Portfolio Payoff Diagram shows the desired payoff. Then, at the left, add the following three assets to your portfolio: one share of stock (long = 1.0), one long put contract with a strike price of $30, and one short call contract with a strike price of $60. You will see that the payoffs to your portfolio exactly match the collar strategy specified in the diagram. You can check the "display" boxes to see the payoff diagram for any one of the portfolio elements or uncheck them all to see only the payoffs for the entire portfolio.

See if you can create portfolios whose payoffs match the other financial strategies listed in the interactive. If you want to become proficient with structured payoffs, set the Financial Strategy to "Random." Whenever you match the payoffs of a portfolio, click "Try Again" to get another random structured payoff "puzzle."

INTERACTIVE ILLUSTRATION 4 Building Structured Payoffs

Scan this QR code, click the image, or use this link to access the interactive illustration: bit.ly/hbsp2pIaFmF



This interactive allows you to create a portfolio of financial instruments and view the combined payoff diagram. Use the input tools below to build the listed financial strategies.

Financial Strategies: Custom

Put-Call Parity: An Example of the Law of One Price

Put-call parity describes the relationship between the price of a European put contract and the price of a European call contract when both contracts are written on the same underlying stock at the same strike price and with the same expiration date. Because both contracts are derived from a common source (the underlying stock), it is not surprising that their prices would be related to each other. The put-call parity relationship makes this relationship explicit and allows the price of a put option to be uniquely (and exactly) determined, given the price of the corresponding call option, and vice versa.

To see this, consider the following example. Assume that you use the techniques described in the previous section to construct a portfolio with the following components and structured payoff:

Portfolio Components

Portfolio Payoffs

$$\text{Portfolio}_T = S_T + \max\left(\$25 - S_T, 0\right) + \min\left(\$25 - S_T, 0\right) = \begin{cases} \$25 \text{ if } S_T \le \$25 \\ \$25 \text{ if } S_T \ge \$25 \end{cases} = \$25$$

The payoff to the portfolio is always $25, regardless of the underlying stock price.

Put-call parity can be seen using Interactive Illustration 4. First, set the Financial Strategy to "Custom." Then create a portfolio of the following three assets: long one share of stock, long one put contract with a strike price of $25, and short one call contract, also with a strike price of $25. Note that the portfolio's payoffs, indicated by the thick gray line on the payoff diagram at the right, are always $25, for all future stock prices. And note that this diagram is the same as the payoff diagram for a long bond in Interactive Illustration 3, when the bond's face value is $25.

By buying a share of stock, buying a put option struck at $25, and selling a call option struck at $25, we can create a portfolio with a guaranteed $25 payoff. In other words, we can create a synthetic security with payoffs identical to those of a riskless, zero-coupon bond with face value equal to the $25 strike price of the put and call options. We know, from the ***law of one price***, that if two assets have identical payoffs, then they must trade for the same price. As a result, the cost of constructing our portfolio must be the same as the cost of purchasing the riskless bond.

Assuming that the underlying stock does not pay dividends over the life of the option contract, the following relationship must hold:

$$S + P - C = \text{PV}\left(X\right)$$

where S is the current stock price, P is the value of a put option with strike price X and expiration date T, and C is the value of a call option with the same strike price X and expiration date T. Because the strike price X is paid in the future, its present value, PV(X), must be calculated to account for the time value of money. By rewriting the above equation, we can express the cost of a put option (or the *put premium*) in terms of the cost of a call option (or the *call premium*):

Put-Call Parity Relationship

$$P + S = C + \text{PV}\left(X\right)$$

The relationship between the put premium and the call premium is known as ***put-call parity***. It shows that the price of a put contract is not determined independently from the price of a call contract. Because the left-hand and right-hand sides of the put-call parity relationship have identical payoffs, without put-call parity, an investor could earn riskless arbitrage profits by creating a long position in the underpriced side of the relationship and a short position in the overpriced side of the relationship. In other words, if the prices for a particular group of put and call contracts were such that $P + S > C + \text{PV}(X)$, then an investor could profit by purchasing the call contract and selling the put contract. As more and more investors engaged in this strategy, the price of the call contract would rise and the price of the put contract would fall until put-call parity was reached. Similarly, if the prices of the put and call contracts were such that $P + S < C + \text{PV}(X)$, then an investor could profit by purchasing the put contract and selling the call contract. As more and more investors engaged in this strategy, the price of the put contract would rise and the price of the call contract would fall until put-call parity was reached.

If the underlying stock is expected to pay dividends over the life of the option contract, then the current stock price (S) will reflect the present value of the dividend payments in addition to the present value of the expected stock payoff at the option maturity date, such that

$$S = \text{PV}(\text{Dividends}) + \text{PV}\left(\text{Expected}\left[S_T\right]\right)$$

For this reason, the present value of the dividend payments expected to be made over the life of the option contract must be subtracted from the current stock price so that it reflects only the present value of the expected stock payoff at the option maturity date. Adjusting for expected dividend payments, the following relationship must hold:

$$\left[S - \text{PV}(\text{Dividends})\right] + P - C = \text{PV}(X)$$

The put-call parity relationship for a dividend-paying stock is then given by

Put-Call Parity Relationship for a Dividend-Paying Stock

$$P + S = C + \text{PV}(X) + \text{PV}(\text{Dividends})$$

2.1.3 Determinants of Option Value

Now that we have developed the relationship between the price of a European call contract and a European put contract, we can discuss the relationship between *option value* and *intrinsic value* and develop an intuitive understanding of the factors that determine the value of an option contract. From our discussion of the basics of financial options, we know that the value of an option contract derives from the value of the underlying security on which it is written. For stock options, the value of a call or put option will depend on the price of the underlying stock (S). Because an option contract provides the right to buy or sell the underlying security at a predetermined price, the value of a call or put option will also depend on its strike price (X). Importantly, because an option contract conveys the right—but not the obligation—to exercise the contract, the value of an option contract is also affected by the amount of uncertainty over future changes in the price of the underlying security, referred to as volatility (σ).[d] Because an option contract conveys rights over a period of time, its value is also affected by the time to expiration (T) and the time value of money (r_f). Finally, an option's value will depend on whether or not the underlying stock is expected to pay dividends during the life of the option contract.

Stock Price and Strike Price

A call option provides the right to *buy* the underlying stock at a fixed price X. Because it is beneficial to buy an asset at a lower price, the value of a call option with a lower exercise price will be greater than the value of a call option with a higher exercise price, all else being equal. Similarly, for a given strike price, the value of a call option will increase as the underlying stock price S increases. When the stock price is much higher than the strike price, the likelihood of a call option being in-the-money at expiration increases, and the value of the call option converges to its intrinsic value, $S - X$. When the stock price is much lower than the strike price, the likelihood of the call option being out-of-the-money at expiration increases, and the value of the call option converges to zero. When a call option is at-the-money, such that $S = X$, and uncertainty over whether the option will be in-the-money or out-of-the-money is greatest, the difference between option value and intrinsic value will be the greatest.

Conversely, a put option provides the right to *sell* the underlying stock at a fixed price X. Therefore, for a given stock price, the value of a put option with a higher exercise price will be greater than the value of a put option with a lower exercise price. Similarly, for a given strike price, the value of a put option will increase as the underlying stock price S decreases. When the stock price is much lower than the strike price, the likelihood of a put option being in-the-money at expiration increases and the value of the put option converges to its intrinsic value, $X - S$. When the stock price is much higher than the strike price, the likelihood of the put option being out-of-the-money at expiration increases and the value of the put option converges to zero. When a put option is at-the-money, such that $S = X$, and uncertainty over

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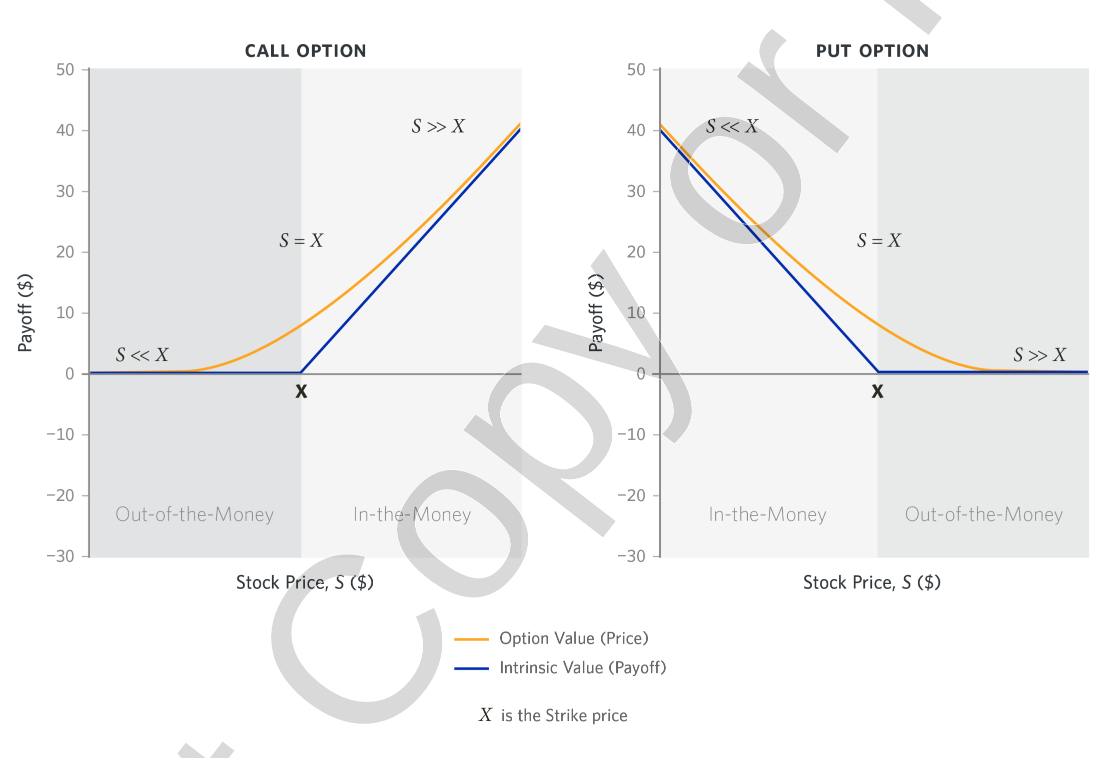
[d] Mathematically, σ refers to the standard deviation of the underlying asset's continuously compounded returns.

whether the option will be in-the-money or out-of-the-money is greatest, the difference between option value and intrinsic value will be the greatest.

Exhibit 3 shows the relationship between option value and intrinsic value for call and put option contracts.

EXHIBIT 3 Option Value Versus Intrinsic Value for Call and Put Options



Volatility of Underlying Asset

For both call and put options, an increase in the volatility of the underlying stock price increases option value. An increase in the volatility of the underlying stock price increases the range in possible future stock price outcomes. Conditional on an option being in-the-money, expected payoffs increase, while out-of-the-money payoffs remain unchanged at zero. For this reason, the value of an option's expected payoff increases as the volatility of the underlying stock price increases, leading to an increase in option value.

Discount Rate

Because an option's strike price is paid or received in the future, the time value of money (or discount rate) also affects the value of call option and put option contracts.

For a call option, an increase in the discount rate increases the option's value. Because the strike price is fixed and is not *paid* until the option is exercised in the future, an increase in the discount rate reduces the present value of the strike price to be paid upon exercise, thus increasing the option's value. For a put option, an increase in the discount rate decreases the option's value. Because the strike price is fixed and is not *received* until the option is exercised in the future, an increase in the discount rate reduces the present value of the strike price to be received upon exercise, thus decreasing the option's value.

Duration of Contract

An option's value is also affected by its time to expiration. For a call option, an increase in the time to expiration will increase an option's value. A longer time to expiration increases the volatility, and the discount rate effects on an option's value. Because both effects are positive, an increase in the time to expiration for a call option increases its value. For a put option, the effect of an increase in the time to expiration cannot be unambiguously determined. Because the volatility component is positive and the risk-free rate component is negative, the effect of an increase in the time to expiration for a put option will depend on the relative magnitude of the two components. If the volatility component is larger than the discount rate component, then an increase in the time to expiration for a put option will increase its value. Otherwise, if the discount rate component is larger than the volatility component, then an increase in the time to expiration for a put option will decrease its value.

Dividends

Finally, an option's value will depend on whether or not the underlying stock is expected to pay dividends during the life of the option contract. As discussed in the section on put-call parity, the payment of dividends (all else equal) will cause a stock's future price to be lower. Thus, the value of a call option written on a dividend-paying stock will be less than the value of a call option written on an equivalent non-dividend-paying stock. Because a call option provides the right to *buy* the underlying stock at a fixed price *X*, the lower expected future stock price implies a *lower* value of a call option contract. For the same reason, the value of a put option written on a dividend-paying stock will be more than the value of a put option written on an equivalent non-dividend-paying stock. Because a put option provides the right to *sell* the underlying stock at a fixed price *X*, the lower expected future stock price implies a *higher* value of a put option contract.

Exhibit 4 summarizes the effect of an increase in each of the six inputs on the Black-Scholes value of call options and put options.

EXHIBIT 4 Effect of an Increase of Each Parameter on Option Value

Input Parameter		Call Option	Put Option
Current stock price	S	*Increases*	*Decreases*
Strike price	X	*Decreases*	*Increases*
Time to expiration	T	*Increases*	*Indeterminate*
Volatility	σ	*Increases*	*Increases*
Risk-free rate	r_f	*Increases*	*Decreases*
Dividend yield	d	*Decreases*	*Increases*

2.1.4 Reinterpreting Corporate Capital Structure

Given the framework for understanding financial options developed in the previous sections, we can apply the framework and the concept of option contracts as claims on future payoffs to develop a deeper understanding of corporate capital structure.

The traditional view of corporate capital structure is that a firm's shareholders (its "owners") have an equity interest in the firm. Shareholders may choose to ***leverage*** the firm by raising additional capital in the form of debt, a claim that is senior to equity. In this way, the firm's shareholders have a "residual claim" on the firm's assets. In other words, the value of the firm's assets (the left-hand side of its balance sheet) is allocated across the various claims against those assets (the right-hand side of its balance sheet), with the most senior claims (debt) being paid first and the most junior claims (equity) being paid last.

In the traditional framework, the value of the firm's equity is found by subtracting the face value of the debt holders' claims from the value of the firm's assets. Equity has a residual interest in the firm in that equity is paid whatever value remains after all of the firm's existing debts are paid. If the value of the firm's assets is not sufficient to repay all of its existing debts, then equity receives nothing. Importantly, because equity has "limited liability," its shareholders are not required to make debt holders "whole" when the value of the firm's assets is less than the value of its obligations. Specifically, if the value of the firm's assets is less than the value of its debts, debt holders will receive less than they are owed and shareholders are *not* required to make up the difference.

Because of the limited liability nature of equity, an options framework can be applied to provide an alternative interpretation of corporate capital structure. In the options interpretation, debt holders "own" the firm's assets and have sold a call option to the firm's shareholders. In other words, when shareholders purchase stock in a firm, they are effectively purchasing a call option. The firm's shareholders have the right, but not the obligation, to purchase the assets of the firm by paying a predetermined strike price, which is the face value of the firm's debts. In this formulation of a firm's capital structure, the debt holders have a short position in a call option contract written on the firm's assets, and the firm's shareholders have the corresponding long position in a call option written on the firm's assets. The firm's shareholders will rationally exercise their call option (by deciding to repay the debt) only if the value of the firm's assets exceeds the face value of the firm's debts.

This interpretation of a firm's capital structure provides an alternative way to determine the value of a firm's equity. Specifically, the value of a firm's equity can be found by determining the value of the implied call option written on the value of the firm's assets. With this interpretation, equity value is equivalent to *option value*, whereas in the traditional interpretation, equity value is equivalent to *intrinsic value*. If the call option is far in-the-money (the value of the firm's assets far exceeds the amount of its debts), then option value will converge to intrinsic value and the two approaches will produce similar results. However, when the call option is *at-the-money* or *out-of-the-money* (the value of the firm's assets are nearly equal to or less than the amount of its debts), the firm is nearing ***financial distress***. In that situation, *option value* will be greater than *intrinsic value* and the two interpretations will produce different results.

The options framework also provides insights on the incentives of the firm's various stakeholders, especially how their incentives differ when a firm nears financial distress. Because equity will exercise their call option when the value of the firm's assets is high, debt holders will not participate in the "upside"; the most they will receive is the face value (exercise price) of their claims. For this reason, debt holders will "own" the underlying firm assets only when the value of the firm's assets is low relative to the firm's obligations. As a result, corporate debt is risky and is equivalent to a risk-free bond plus a short position in a put option written on the value of the firm's assets. For this reason, debt holders tend to be conservative and maximize the value of their claim by avoiding risk (i.e., minimizing the value of the short put and the short call position that they hold). In contrast, shareholders tend to be aggressive and maximize the value of their claim by seeking out risk (i.e., maximizing the value of the long call position that they hold).

2.2 Option Valuation

In this section, we develop the techniques used to determine the value of an option contract. Specifically, we apply the concepts of payoffs and ***replicating portfolios*** to the valuation of options. The underlying idea is simple but very powerful. We will derive the value of an option contract by combining assets with known values to create a portfolio that replicates the option's payoffs. In accordance with the *law of one price*, the value of the option contract must be equal to the value of the replicating portfolio. Because we know the values of the assets used to create the replicating portfolio, we can calculate the value of the replicating portfolio––and thus the value of the option. In other words, we can think of an option contract as having been "constructed" by an option writer who creates a portfolio of assets with a combined payoff function that is identical to the option's payoff function. In a competitive market, the price that the writer receives for selling the option contract will equal the cost of constructing the portfolio that generates the option's payoffs. This is the concept behind "replication."

The section is organized as follows. We first develop the intuition for the analysis in the context of a discrete, two-payout example.[e] We then apply this intuition to understand the solution to the more general situation in which there is a continuum of possible future payouts, commonly known as the *Black-Scholes option pricing formula*. Throughout the first part of the reading, we will focus on the valuation of European call options written on stocks that do not pay dividends. At the end of this section, we discuss the valuation of European options written on stocks that do pay dividends.

2.2.1 Replicating the Payoffs to an Option Contract

Assume you are considering the purchase of a one-year, European call option written on the stock of the Simple Solutions Corporation. The option has a strike price of $40, and the current Simple Solutions stock price is $35. Simple Solutions is an interesting (and unique) company, because its stock price in one year will equal one of only two values: the future stock price will be either $60 or $20. To decide how much you might be willing to pay for the call option, you need to know what the option is worth.

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[e] The ideas presented in the following sections date to J. Cox, S. Ross, and M. Rubinstein, "Option Pricing: A Simplified Approach," *Journal of Financial Economics* 7 (1979): 229–263, which developed the discrete-time approach to pricing option contracts.

To determine the value of the Simple Solutions call option, we need to understand its potential future payoffs. Because there are only two potential payoffs to the underlying stock, there are only two potential payoffs to the option contract. If the future stock price equals \$60, the payout to the option contract will equal \$20 (the \$60 stock price less the \$40 strike price). On the other hand, if the future stock price equals only \$20, the payout to the option contract will equal \$0 because the option will not be exercised. On the expiration date T, the Simple Solutions call option has the following payoff function:

$$C_T = \begin{cases} \max\left(\$60-\$40, 0\right) = \$20 & \text{if } S_T = \$60 \\ \max\left(\$20-\$40, 0\right) = \$0 & \text{if } S_T = \$20 \end{cases}$$

Given the two potential call option payoffs (\$20 and \$0), we can use the underlying stock and a bond to create a portfolio with payoffs that are identical to those of the call option. Assume that you borrow \$10 (a short position in a bond) and use the money to buy 0.5 shares of Simple Solutions stock. You now have a portfolio with the following positions:

Portfolio Components

Long 0.5 shares of Simple Solutions stock + Short \$10 bond

Because you will need to repay the \$10 you borrowed (assuming a zero interest rate) regardless of the realized future value of the stock price, the payoffs to the portfolio are

Replicating Portfolio Payoffs

$$\text{Portfolio}_T = \begin{cases} 0.5 \times \$60 - \$10 = \$20 & \text{if } S_T = \$60 \\ 0.5 \times \$20 - \$10 = \$0 & \text{if } S_T = \$20 \end{cases}$$

The payoffs to the constructed portfolio exactly replicate the payoffs to the call option contract. Because the payoffs from the replicating portfolio are identical to the payoffs for the call option, the law of one price tells us that the value of the call option must be the same as the value of the replicating portfolio. Therefore, the current value of the call option (C) must equal the current value of the replicating portfolio, which is

$$0.5 \times \$35 - \$10 = \$7.5$$

Having found the value of the call option contract, we can also use the put-call parity relationship from the previous section to find the value of the corresponding put option contract. From put-call parity, we know that

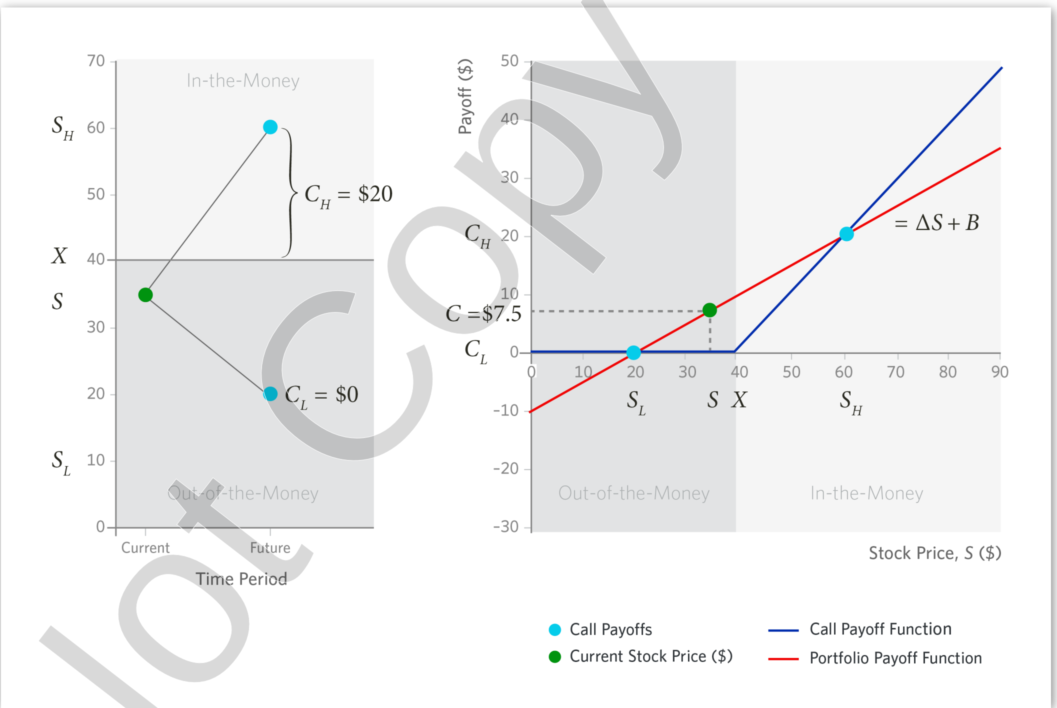
$$S + P = \mathrm{PV}(X) + C$$

Therefore,

$$P = \mathrm{PV}(X) - S + C = \$40 - \$35 + \$7.5 = \$12.5$$

Exhibit 5 depicts the discrete two-payout call option example described above.

EXHIBIT 5 Replicating Call Payoffs in the Two-Payout Example



The technique illustrated in the previous example is not dependent on the particular parameters that were chosen and can easily be made more general. It can be adapted to allow for different values of the stock price and strike price, different future stock prices (with a different range in values), a non-zero interest rate, and an arbitrary time to expiration.

To generalize the model, assume that the future Simple Solutions stock price (S_T) will either be high (S_H) or low (S_L) at some future time T. Then, the future payoff to the Simple Solutions call option is either C_H or C_L:

$$C_T = \begin{cases} C_H = S_H - X & \text{if } S_T = S_H \\ C_L = 0 & \text{if } S_T = S_L \end{cases}$$

By holding a short position B in a bond (borrowing) and using the money to purchase Δ (Greek letter *delta*) shares of Simple Solutions stock (a long position), we can construct a portfolio with payoffs that replicates the payoffs to the call option contract:

$$\text{Payoff}_T = \begin{cases} C_H = \Delta S_H + B & \text{if } S_T = S_H \\ C_L = \Delta S_L + B & \text{if } S_T = S_L \end{cases}$$

Because we have two equations and two parameters, we can solve for Δ and B:

$$\Delta = \frac{C_H - C_L}{S_H - S_L} \text{ and } B = C_L - \Delta S_L$$

The parameter Δ, called the ***option delta***, measures the sensitivity of the option's value to changes in the price of the underlying asset. The parameter B is the short position in the bond used to construct the replicating portfolio. Finally, the value of the option must equal the value of the replicating portfolio, such that the price of the call option is given by the following equation:

$$C = S\Delta + \text{PV}(B)$$

To see that this more general expression is consistent with our numerical example, we can calculate the values of Δ and B and check that the implied value of the call option matches our original answer. In the numerical example,

$$\Delta = \frac{C_H - C_L}{S_H - S_L} = \frac{\$20 - \$0}{\$60 - \$20} = 0.5$$

and

$$B = C_L - \Delta S_L = \$0 - 0.5 \times \$20 = -\$10$$

We can then substitute the values of Δ and B into the above equation to get the corresponding value of the call option:

$$C = \$35\Delta + \text{PV}(B) = \$35 \times 0.5 - \$10 = \$7.5$$

This value equals the value we obtained in our numerical example and confirms that the more general option pricing equation is consistent with our original example.

The technique of replicating the payoffs to an option contract can also be used to determine the value of a put option contract. Just as in the previous example, we can determine the payoffs to a put option and then replicate the payoffs by constructing a portfolio consisting of a position in the underlying stock and a position in a risk-free bond. To differentiate between the replicating portfolio for a call option contract and that for a put option contract, we will label the portfolio parameters for the call option contract as Δ_{Call} and B_{Call} and the parameters for the put option contract as Δ_{Put} and B_{Put}. We can use the put-call parity relationship to verify that the parameters for the two portfolios will always be such that

$$\Delta_{\text{Put}} = \Delta_{\text{Call}} - 1$$

and

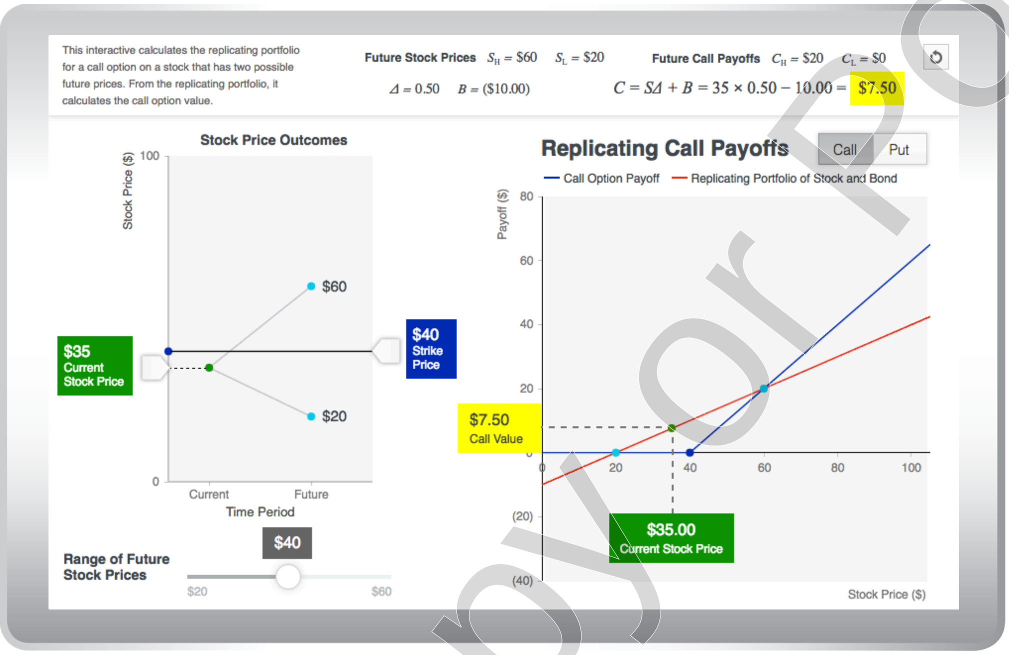
$$B_{\text{Put}} = X + B_{\text{Call}}$$

Interactive Illustration 5 allows you to explore the relationship between an option contract and its replicating portfolio, as well as the relationship between the respective call option and put option parameters.

The interactive's left graph shows the current stock price and the two possible future stock prices, S_H and S_L. On the right graph, the dark blue line depicts the payoff function of a call option with the specified strike price. The light blue dots on the line indicate the call payoffs, C_H and C_L, for the two possible future prices of the underlying stock. The red line depicts the payoff function for the replicating portfolio, which is constructed by calculating Δ_{Call} and B_{Call} to have the same payoffs as the call option at S_H and S_L. Try adjusting the current stock price, the call option's strike price, and the distance between S_H and S_L. For every set of S_H and S_L, the interactive calculates the values of Δ_{Call} and B_{Call} and, from those two parameters, the value of the call option, C. You can also click the toggle button at the top right of the interactive to see the value of the put option (P) and its parameters Δ_{Put} and B_{Put}.

INTERACTIVE ILLUSTRATION 5 Replicating Portfolio for Two Payoffs

Scan this QR code, click the image, or use this link to access the interactive illustration: bit.ly/hbsp2IV9Xuh



This interactive calculates the replicating portfolio for a call option on a stock that has two possible future prices. From the replicating portfolio, it calculates the call option value.

Future Stock Prices $S_H = \$60$ $S_L = \$20$

$\Delta = 0.50$ $B = (\$10.00)$

Future Call Payoffs $C_H = \$20$ $C_L = \$0$

$C = S\Delta + B = 35 \times 0.50 - 10.00 = \$7.50$

In summary, the equations for Δ and *B* for call and put options are

Call	$\Delta_{\text{Call}} = \dfrac{C_H - C_L}{S_H - S_L}$	$B_{\text{Call}} = C_L - \Delta_{\text{Call}} S_L$
Put	$\Delta_{\text{Put}} = \dfrac{P_H - P_L}{S_H - S_L}$	$B_{\text{Put}} = P_L - \Delta_{\text{Put}} S_L$

In this discussion, we have used a single-period, two-payout example to introduce and develop the key concepts used to value option contracts. We have shown that, in a competitive market, the value of an option contract must equal the cost of constructing a portfolio that replicates the payoffs to the option contract. Clearly, our example model is overly simplistic: Stock prices (and option payoffs) take on more than just two values, and prices adjust continually throughout the life of the contract, not just at expiration. However, the intuition and concepts that the model highlights are fundamental and crucial to all option pricing theory. In the next section, we will expand the model to allow for multiple payouts and rebalancing of the replicating portfolio during the life of the option.

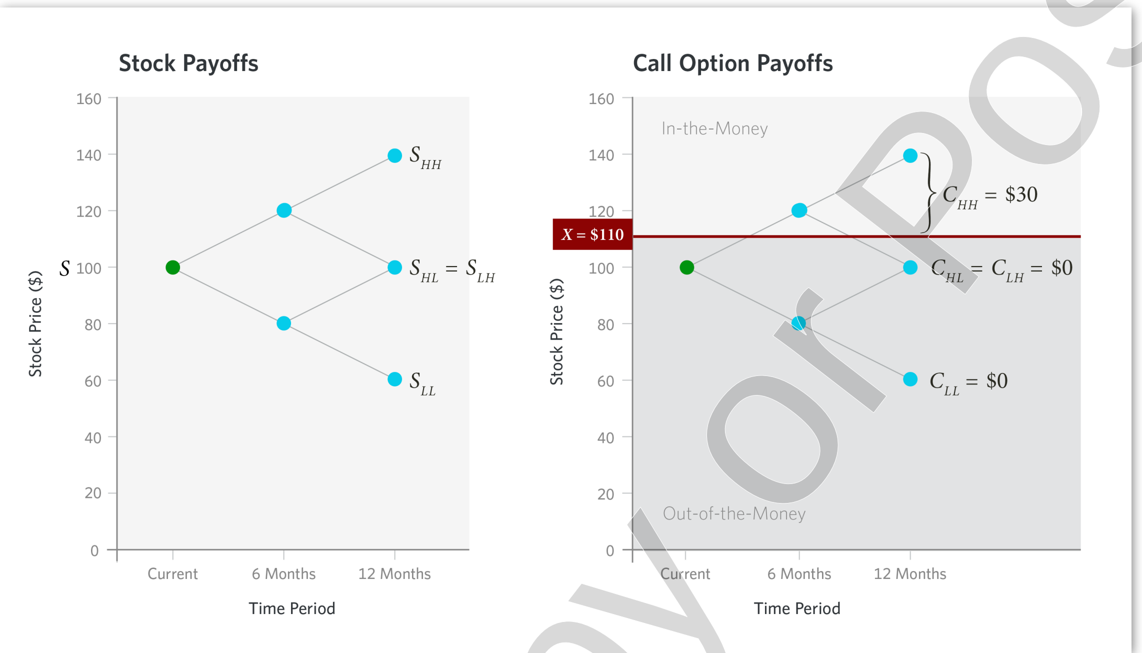
2.2.2 Dynamic Replication

In this section, we introduce the concept of *dynamic replication*. Because the price of the underlying stock changes during the life of the option contract, the portfolio that replicates the payoffs to the option contract also changes. As such, the values of Δ and B that replicate an option's payoffs change as the underlying stock price evolves over time. For this reason, the replicating portfolio must be continuously rebalanced. This is referred to as *dynamic replication*. To see how dynamic replication works, we will first construct a two-period example with three possible future payoffs. We will then introduce the Black-Scholes option pricing formula, which solves for the values of Δ and B when the replicating portfolio is adjusted continuously.

Assume you are considering the purchase of a one-year, European call option written on the stock of Dynamic Enterprises, Inc. The option has a strike price of \$110, and the current Dynamic Enterprises stock price is \$100. Dynamic Enterprises is (somewhat) more interesting than Simple Solutions because its stock price will adjust every six months instead of just once at the end of the year. Specifically, the Dynamic Enterprises stock price will increase or decrease by \$20 each six-month period. Therefore, the Dynamic Enterprises stock price will be either \$120 (S_H) or \$80 (S_L) after six months and \$140 (S_{HH}), \$100 ($S_{HL} = S_{LH}$), or \$60 (S_{LL}) after 12 months.

The option's payoffs will be realized at the end of the 12-month period. If the Dynamic Enterprises stock price is higher than the \$110 strike price, then the payoff will be positive. Specifically, if the stock price in 12 months is \$140, then the payoff to the call option will equal \$30 (= \$140 – \$110). If the Dynamic Enterprises stock price is lower than the \$110 strike price, then the payoff will be zero. We want to know what the option is worth when issued. (Stock prices and call option payoffs are depicted in **Exhibit 6**.)

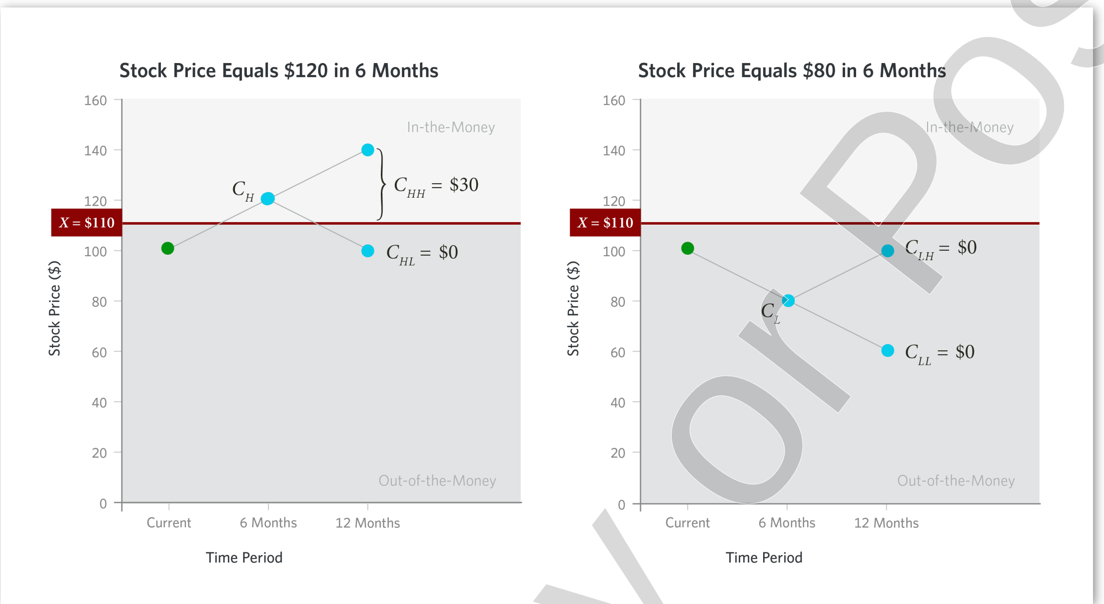
EXHIBIT 6 Stock and Call Payoffs in the Two-Period Example



To determine the value of the Dynamic Enterprises call option, we need to construct a portfolio that will replicate its potential future payoffs. However, because the Dynamic Enterprises stock price changes every six months, we will be able to adjust the composition of the portfolio after we observe the six-month change in the Dynamic enterprises stock price. The fact that we will be able to rebalance the replicating portfolio in the future makes the problem dynamic. For this reason, we first determine the composition of the replicating portfolio (and the associated value of the call option) for each possible six-month change in the Dynamic Enterprises stock price. We then construct the current portfolio that generates payoffs in six months that are equal to the values of the two potential future replicating portfolios.

Exhibit 7 shows the call option payoffs conditional on the realized value of the Dynamic Enterprises stock price after six months. The left-hand panel shows the call option payoffs conditional on the stock price increasing to $120 after six months, and the right-hand panel shows the call option payoffs conditional on the stock price decreasing to $80 after six months.

EXHIBIT 7 Conditional Call Payoffs in the Two-Period Example



First, assuming that the stock price has increased to $120 after six months, we construct the replicating portfolio. The stock price after 12 months will be either $140 or $100, and the payoffs to the call option C_H will equal $30 or $0, respectively. Therefore, for C_H, the parameters for the replicating portfolio (Δ_H and B_H) are

Replicating Portfolio Parameters: Stock Price Equals $120 After Six Months

$$\Delta_H = \frac{C_{HH} - C_{HL}}{S_{HH} - S_{HL}} = \frac{\$30 - \$0}{\$140 - \$100} = 0.75$$

and

$$B_H = C_{HL} - \Delta_H S_{HL} = \$0 - 0.75 \times \$100 = -\$75$$

The value of the call option C_H is equal to the cost to construct the replicating portfolio:

Call Option Value: Stock Price Equals $120 After Six Months

$$C_H = S_H \Delta_H + \text{PV}\left(B_H\right) = \$120 \times 0.75 - \$75 = \$15$$

In similar fashion, assuming that the stock price has decreased to $80 after six months, we construct the replicating portfolio for C_L. In this case, the stock price after 12 months will be either $100 or $60. Either way, the call option contract will be out-of-the-money and its payoffs will be zero. Given these conditional payoffs, the parameters for the replicating portfolio (Δ_L and B_L) are

Replicating Portfolio Parameters: Stock Price Equals $80 After Six Months

$$\Delta_L = \frac{C_{LH} - C_{LL}}{S_{LH} - S_{LL}} = \frac{\$0 - \$0}{\$100 - \$60} = 0.0$$

and

$$B_L = C_{LL} - \Delta_L S_{LL} = \$0 - 0.0 \times \$60 = \$0$$

The value of the call option C_L is equal to the cost to construct the replicating portfolio:

Call Option Value: Stock Price Equals $80 After Six Months

$$C_L = S_L \Delta_L + \mathrm{PV}\left(B_L\right) = \$80 \times 0.0 + \$0 = \$0$$

Given the two conditional values (C_H and C_L) for the call option value after six months, we can then construct the replicating portfolio for C:

Replicating Portfolio Parameters: Current Stock Price

$$\Delta = \frac{C_H - C_L}{S_H - S_L} = \frac{\$15 - \$0}{\$120 - \$80} = 0.375$$

and

$$B = C_L - \Delta S_L = \$0 - 0.375 \times \$80 = -\$30$$

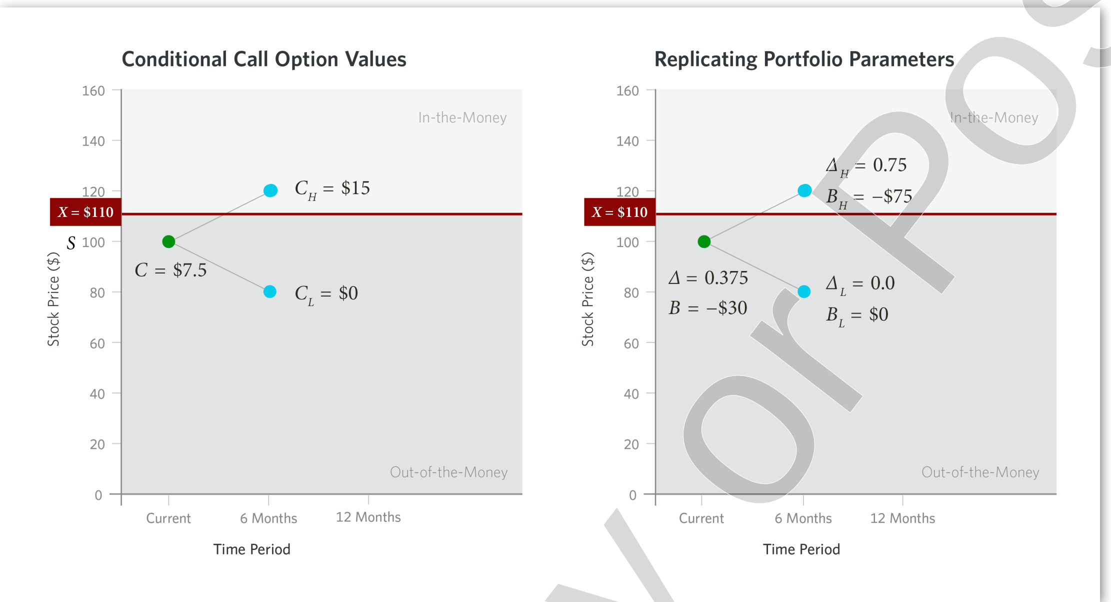
Finally, the current value of the call option is given by

Call Option Value: Current Stock Price

$$C = S\Delta + \mathrm{PV}\left(B\right) = \$100 \times 0.375 - \$30 = \$7.5$$

Exhibit 8 shows the conditional call option values and the parameters for the corresponding replicating portfolios.

EXHIBIT 8 Conditional Call Option Values and Replicating Portfolios



The Dynamic Enterprises example shows how the final call option payoffs can be generated by dynamically adjusting (rebalancing) the replicating portfolio over time. Although this example is more realistic than the Simple Solutions example, it is still quite simplistic. In reality, stock prices adjust over very short periods of time and the replicating portfolio must be frequently rebalanced. Given the algebra involved in our two examples, you might think that it would be difficult to determine the value of an option contract when the price of the underlying stock is changing continuously. Fortunately, there is an equation that makes it easy to determine the parameters for the replicating portfolio when prices change continuously over the life of an option contract.

2.2.3 The Black-Scholes-Merton Option Pricing Formula

In their seminal work from the early 1970s, Fisher Black, Myron Scholes, and Robert Merton derived a formula that solves for the parameters of the replicating portfolio at each point in time during the life of a European option contract.[f] Commonly referred to as the Black-Scholes formula, the resulting equation revolutionized the valuation of financial options and has become widely adopted as the standard method for determining the value of European options. In this section, we present the Black-

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[f] See F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81, no. 3 (1973): 637–654, and R. Merton, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science* 4 (Spring 1973): 141–183.

Scholes formula, discuss its properties, and show how and when to use it to value an option contract.

Understanding the Formula

Although Black, Scholes, and Merton originally used a different technique, the Black-Scholes formula can be derived from the discrete two-payoff model previously discussed. As the time interval during which the underlying stock price changes becomes small, the amount by which the underlying stock price is likely to change also becomes small. For this reason, as the time interval goes to zero, the discrete model will have an infinite number of instantaneous time periods, during which the stock price can be thought of as taking on one of only two values. In the limit, the resulting stock price changes become continuous. The distribution of *stock prices* converges to the lognormal distribution, and the distribution of *stock returns* converges to the normal distribution. (For more on the distributions of stock prices and returns, see *Core Reading: Risk and Return 1: Stock Returns and Diversification* [HBP No. 5220.])

The Black-Scholes analysis makes the following assumptions: lognormal price distribution of the underlying security, continuous trading of the underlying security, no transactions costs or other frictions (including no limits on short positions), constant volatility of the underlying security, and a constant risk-free rate. Given these assumptions, Black, Scholes, and Merton used the concept of dynamic replication to solve for the value of a European call option on a non-dividend-paying stock. Their solution gives the value of a call option as a function of the current stock price S, the option's strike price X, the volatility of the underlying stock price σ, the time to expiration of the option T, and the risk-free interest rate r. It is described by the following formula:

Black-Scholes Option Pricing Formula

$$C = S\mathrm{N}(d_1) - Xe^{-rT}\mathrm{N}(d_2)$$

where

$$d_1 = \frac{ln(S/X) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

The function $\mathrm{N}(x)$ is the ***cumulative standard normal distribution***, the cumulative probability function for a standard normal variable (mean zero and standard deviation of one). Since they are probabilities, the values of both $\mathrm{N}(d_1)$ and $\mathrm{N}(d_2)$ are between 0 and 1. Although the formula may look complicated at first glance, a closer look reveals that it is simply the portfolio that replicates the payoffs to the corresponding call option. Analogous to our previous discrete examples, the function $\mathrm{N}(d_1)$ is the option delta and describes the long position in the underlying stock, and the term $Xe^{-rT}\mathrm{N}(d_2)$ gives the magnitude of the short bond position used to finance the long stock position. (You may remember from the time value of money that e^{-rT} is the formula for continuous discounting.) In other words, the Black-Scholes value for a call option is the cost to construct the corresponding replicating portfolio with parameters:

Black-Scholes Replicating Portfolio Parameters

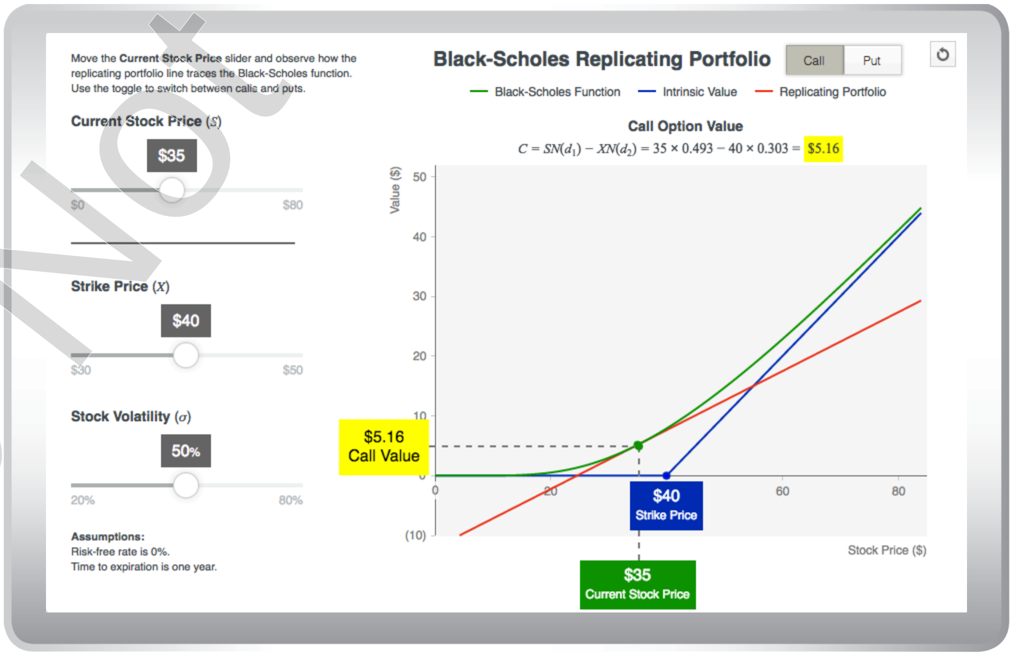
$$\Delta = \mathrm{N}\left(d_1\right)$$

$$B = -Xe^{-rT}\mathrm{N}\left(d_2\right)$$

Interactive Illustration 6 is similar to Interactive Illustration 5, but it uses the Black-Scholes formula to calculate the replicating portfolio. It shows the replicating portfolio at one point in time for a given value of the underlying stock price. By varying the price of the underlying stock, you can trace out the value of the option contract and compare it to the corresponding payoff function for the contract.

INTERACTIVE ILLUSTRATION 6 Black-Scholes Replicating Portfolio

Scan this QR code, click the image, or use this link to access the interactive illustration: bit.ly/hbsp2GgcfCD



Move the **Current Stock Price** slider and observe how the replicating portfolio line traces the Black-Scholes function. Use the toggle to switch between calls and puts.

Current Stock Price (S): $35

Strike Price (X): $40

Stock Volatility (σ): 50%

Assumptions:
Risk-free rate is 0%.
Time to expiration is one year.

Call Option Value

$C = SN(d_1) - XN(d_2) = 35 \times 0.493 - 40 \times 0.303 =$ $5.16

A Numerical Example

In this section, we demonstrate how to use the Black-Scholes formula and reinforce the concepts that have been developed. To highlight the effect of each of the inputs on the Black-Scholes value, the example is presented in four steps. First, we assume that the risk-free interest rate is zero so that there is no discounting and apply the Black-Scholes methodology to an at-the-money option. Then, we analyze the same option assuming that the stock price is higher, such that the option is in-the-money. Next, we assume that the risk-free interest rate is positive, such that the exercise price needs to be discounted. And, finally, we assume a longer time to expiration, such that there is a longer period of time over which the underlying stock price can evolve. The example provides the opportunity to map the specific Black-Scholes parameters to the concepts we have developed in the previous sections and to highlight the effect of the various input parameters on call option and put option values.

Assume you are considering the purchase of a one-year, at-the-money, European call option written on the common stock of the Continuous Media Corporation. To decide how much you might be willing to pay for the option, you want to know its Black-Scholes value. The current Continuous Media stock price is \$35, and the volatility of its stock price is 40% per year. In addition, Continuous Media does not expect to make any dividend payments over the next year. Because the option is struck at-the-money, the exercise price equals the current \$35 stock price. Finally, to keep the calculations simple, assume that the risk-free interest rate is zero. Note that the time period of the parameters must match: In this case, the duration of the contract is one year and the numbers for the risk-free rate and the volatility are in terms of years.

Because the risk-free rate is equal to zero, there is no discounting and the replicating portfolio is given by

$$C = S\mathrm{N}(d_1) - X\mathrm{N}(d_2)$$

The option delta, N(d_1), is found by calculating the value of d_1. Because the option is struck at-the-money, *ln*(*S*/*X*) = 0, and the expression for d_1 simplifies to

$$d_1 = \frac{0.5\sigma^2 T}{\sigma\sqrt{T}} = 0.5\sigma\sqrt{T} = 0.5 \times 0.4 \times \sqrt{1} = 0.2$$

The cumulative probability for 0.2 on a standard normal distribution is

$$\mathrm{N}(d_1) = \mathrm{N}(0.2) = 0.579$$

The portion of the strike price that is financed by borrowing, N(d_2), is found by calculating the value of d_2:

$$d_2 = d_1 - \sigma\sqrt{T} = 0.2 - 0.4 \times \sqrt{1} = -0.2$$

and

$$\mathrm{N}\left(d_2\right) = \mathrm{N}\left(-0.2\right) = 0.421$$

Given the values of N(d_1) and N(d_2), the cost of the replicating portfolio and the corresponding value of the call option are given by the formula

Black-Scholes Call Option Value

$$C = S\mathrm{N}\left(d_1\right) - X\mathrm{N}\left(d_2\right) = \$35 \times 0.579 - \$35 \times 0.421 = \mathbf{\$5.55}$$

Finally, the Black-Scholes value of the corresponding put option can be found by using the put-call parity relationship, such that

Black-Scholes Put Option Value

$$P = \mathrm{PV}\left(X\right) - S + C = \$35 - \$35 + \$5.55 = \mathbf{\$5.55}$$

It makes sense that the put and call contracts have the same value, since both are struck at-the-money and the risk-free rate equals zero.

Exhibit 9 summarizes the Black-Scholes valuation of the Continuous Media Enterprises option contracts based on the previous assumptions. The top panel of the exhibit lists the five input assumptions and shows the intermediate calculations described above. Panel A uses the intermediate calculations to construct the replicating portfolio and the corresponding value of the call option. Panel B uses the put-call parity relationship together with the calculated call option value to determine the corresponding value of the put option.

EXHIBIT 9 Black-Scholes Value of Call and Put Options for Continuous Media Enterprises

Input Assumptions		**Intermediate Calculations**	
S	\$35.00	PV(X)	\$35.00
X	\$35.00	d_1	0.200
T	1.00	d_2	(0.200)
σ	40.0%	N(d_1)	0.579
r_f	0.0%	N(d_2)	0.421

PANEL A

Calculation of Black-Scholes Call Option Value

Stock price (S)			\$35.00
Option delta (Δ)	[= N(d_1)]	×	0.58
ΔS			\$20.27
+PV(B)	[= −N(d_2) × PV(X)]		(\$14.73)
= Black-Scholes Call Option Value (*C*)			**\$5.55**

PANEL B

Calculation of Black-Scholes Put Option Value

PV(X)	\$35.00
+ *B-S* Call Option Value	\$5.55
− Stock price	(\$35.00)
= Black-Scholes Put Option Value (*P*)	**\$5.55**

Now assume that the Continuous Media stock price increases to \$40, such that the call option is in-the-money by \$5 and the put option is out-of-the-money by \$5. To determine the new Black-Scholes values for the options, we must calculate new values for d_1 and d_2 and construct a new replicating portfolio. The value of d_1 and N(d_1) are now given by

$$d_1 = \frac{ln\left(S/X\right)+0.5\sigma^2 T}{\sigma\sqrt{T}} = \frac{ln\left(\$40/\$35\right)+0.5\times 0.4^2\times 1}{0.4\times\sqrt{1}} = 0.534$$

and

$$\mathrm{N}\left(d_1\right) = \mathrm{N}\left(0.534\right) = 0.703$$

Similarly, the corresponding values of d_2 and N(d_2) are given by

$$d_2 = d_1 - \sigma\sqrt{T} = 0.534 - 0.4 \times \sqrt{1} = 0.134$$

and

$$\mathrm{N}(d_2) = \mathrm{N}(0.134) = 0.553$$

Given the values of N(d_1) and N(d_2), the Black-Scholes values of the call and put options are given by

Black-Scholes Call and Put Option Values

$$C = S\mathrm{N}(d_1) - X\mathrm{N}(d_2) = \$40 \times 0.703 - \$35 \times 0.553 = \mathbf{\$8.77}$$

and

$$P = \mathrm{PV}(X) - S + C = \$35 - \$40 + \$8.77 = \mathbf{\$3.77}$$

This shows that an increase in the stock price results in an increase in the value of the call option and a decrease in the value of the corresponding put option. In addition, because the risk-free interest rate equals zero and there is no discounting, the difference between the values of the call and put options reflects the spread between the stock price and the option strike price. **Exhibit 10** summarizes the Black-Scholes valuation of the Continuous Media Enterprises option contracts assuming a higher $40 stock price.

EXHIBIT 10 Black-Scholes Value with $40 Stock Price

Input Assumptions		**Intermediate Calculations**	
S	$40.00	PV(X)	$35.00
X	$35.00	d_1	0.534
T	1.00	d_2	0.134
σ	40.0%	N(d_1)	0.703
r_f	0.0%	N(d_2)	0.553

PANEL A
Calculation of Black-Scholes Call Option Value

Stock price (S)			$40.00
Option delta (Δ)	[= N(d_1)]	×	0.70
ΔS			$28.13
+ PV(B)	[= −N(d_2) × PV(X)]		($19.36)
= Black-Scholes Call Option Value (*C*)			**$8.77**

PANEL B
Calculation of Black-Scholes Put Option Value

PV(X)	$35.00
+ *B-S* Call Option Value	$8.77
– Stock price	($40.00)
= Black-Scholes Put Option Value (*P*)	**$3.77**

Next, assume that the continuously compounded risk-free interest rate increases to 3%. To determine the new Black-Scholes values for the options, we must again calculate new values for d_1 and d_2 and construct a new replicating portfolio. The value of d_1 and N(d_1) are now given by

$$d_1 = \frac{ln\left(S/X\right)+\left(r+0.5\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{ln\left(\$40/\$35\right)+\left(0.03+0.5\times0.4^2\right)\times1}{0.4\times\sqrt{1}} = 0.609$$

and

$$\mathrm{N}\left(d_1\right)=\mathrm{N}\left(0.609\right)=0.729$$

Similarly, the corresponding values of d_2 and N(d_2) are given by

$$d_2 = d_1 - \sigma\sqrt{T} = 0.609 - 0.4 \times \sqrt{1} = 0.209$$

and

$$\mathrm{N}\left(d_2\right) = \mathrm{N}\left(0.209\right) = 0.583$$

Given the values of N(d_1) and N(d_2), the Black-Scholes values of the call and put options are given by

Black-Scholes Call and Put Option Values

$$C = S\mathrm{N}\left(d_1\right) - Xe^{-rT}\mathrm{N}\left(d_2\right) = \$40 \times 0.729 - \$35 \times e^{-0.03} \times 0.583 = \mathbf{\$9.36}$$

and

$$P = Xe^{-rT} - S + C = \$35 \times e^{-0.03} - \$40 + \$9.36 = \mathbf{\$3.32}$$

In this example, an increase in the continuously compounded risk-free interest rate from 0% to 3% results in an increase in the value of the call option from \$8.77 to \$9.36 and a decrease in the value of the put option from \$3.77 to \$3.32. The value of the call option increases because the exercise price is not paid until the option expires in one year. Similarly, the value of the put option decreases because the exercise price is not received until the option expires in one year. The difference between the value of the call option and the put option reflects the spread between the stock price and the discounted value of the strike price. **Exhibit 11** summarizes the Black-Scholes valuation assuming a \$40 stock price and a 3% risk-free rate.

EXHIBIT 11 Black-Scholes Value with a $40 Stock Price and 3% Risk-Free Interest Rate

Input Assumptions		**Intermediate Calculations**	
S	$40.00	PV(X)	$33.97
X	$35.00	d_1	0.609
T	1.00	d_2	0.209
σ	40.0%	N(d_1)	0.729
r_f	3.0%	N(d_2)	0.583

PANEL A
Calculation of Black-Scholes Call Option Value

Stock price (S)			$40.00
Option delta (Δ)	[= N(d_1)]	×	0.73
ΔS			$29.15
+ PV(B)	[= −N(d_2) × PV(X)]		($19.79)
= Black-Scholes Call Option Value (*C*)			**$9.36**

PANEL B
Calculation of Black-Scholes Put Option Value

PV(X)	$33.97
+ *B-S* Call Option Value	$9.36
− Stock price	($40.00)
= Black-Scholes Put Option Value (*P*)	**$3.32**

Finally, assume that the time to expiration of the option contract is increased from one year to two years. To determine the new Black-Scholes values for the options, we must again calculate new values for d_1 and d_2 and construct a new replicating portfolio. The value of d_1 and N(d_1) are now given by

$$d_1 = \frac{ln\left(S/X\right)+\left(r+0.5\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{ln\left(\$40/\$35\right)+\left(0.03+0.5\times0.4^2\right)\times 2}{0.4\times\sqrt{2}} = 0.625$$

and

$$\mathrm{N}\left(d_1\right)=\mathrm{N}\left(0.625\right)=0.734$$

Similarly, the corresponding values of d_2 and $\mathrm{N}(d_2)$ are given by

$$d_2 = d_1 - \sigma\sqrt{T} = 0.625 - 0.4 \times \sqrt{2} = 0.059$$

and

$$\mathrm{N}\left(d_2\right) = \mathrm{N}\left(0.059\right) = 0.524$$

Given the values of $\mathrm{N}(d_1)$ and $\mathrm{N}(d_2)$, the Black-Scholes values of the call and put options are given by

Black-Scholes Call and Put Option Values

$$C = S\mathrm{N}\left(d_1\right) - Xe^{-rT}\mathrm{N}\left(d_2\right) = \$40 \times 0.734 - \$35 \times e^{-0.03\times 2} \times 0.524 = \mathbf{\$12.10}$$

and

$$P = Xe^{-rT} - S + C = \$35 \times e^{-0.03\times 2} - \$40 + \$12.10 = \mathbf{\$5.06}$$

In this last example, an increase in length of time to expiration results in an increase in the value of the call option from $9.36 to $12.10 and an increase in the value of the put option from $3.32 to $5.06. The value of the call option increases because the longer time to expiration allows for a greater range of potential future stock prices and option payoffs. In addition, the time to expiration also results in greater discounting of the exercise price. Because of the greater range of potential future payoffs, the value of the put option also increases in value, even though the exercise price is not received for two years and is more heavily discounted. In this particular example, the increase in the value of the put option resulting from greater volatility is more than the reduction in value resulting from greater discounting. **Exhibit 12** summarizes the Black-Scholes valuation assuming an increase in the time to expiration from one year to two years.

EXHIBIT 12 Black-Scholes Value with Expiration in Two Years

Input Assumptions		**Intermediate Calculations**	
S	\$40.00	PV(X)	\$32.96
X	\$35.00	d_1	0.625
T	2.00	d_2	0.059
σ	40.0%	N(d_1)	0.734
r_f	3.0%	N(d_2)	0.524

PANEL A
Calculation of Black-Scholes Call Option Value

Stock price (S)			\$40.00
Option delta (Δ)	[= N(d_1)]	×	0.73
ΔS			\$29.36
+PV(B)	[= −N(d_2) × PV(X)]		(\$17.26)
= Black-Scholes Call Option Value (*C*)			**\$12.10**

PANEL B
Calculation of Black-Scholes Put Option Value

PV(X)	\$32.96
+ *B-S* Call Option Value	\$12.10
− Stock price	(\$40.00)
= Black-Scholes Put Option Value (*P*)	**\$5.06**

Incorporating Dividends in the Formula

In this section, we show how to use the Black-Scholes formula to value an option written on a dividend-paying stock. In the Black-Scholes formula discussed previously, we assumed that the underlying stock did not pay dividends, such that the current stock price was S = PV(Expected[S_T]). If the underlying stock is expected to pay dividends over the life of the option contract, then the current stock price (S) will reflect the present value of the dividend payments in addition to the present value of the expected stock payoff at the option maturity date, so that

$$S = \text{PV}\left(\text{Dividends}\right) + \text{PV}\left(\text{Expected}\left[S_T\right]\right)$$

Because the Black-Scholes formula assumes that

$$S = \mathrm{PV}\left(\text{Expected}\left[S_T\right]\right)$$

the present value of the dividend payments expected to be made over the life of the option contract must be subtracted from the current stock price so that it reflects only the present value of the expected stock payoff at the option maturity date. To determine the Black-Scholes value of a dividend-paying stock, we first construct an adjusted stock price (S^*), such that

$$S^* = \mathrm{PV}\left(\text{Expected}\left[S_T\right]\right) = S - \mathrm{PV}\left(\text{Dividends}\right)$$

For the special case in which the underlying stock pays a continuous dividend yield (d), Merton[g] showed that

$$S^* = Se^{-dT}$$

The adjusted stock price (S^*) is then used in the Black-Scholes formula in place of the actual stock price (S).[h] This yields the following adjusted Black-Scholes formula:

Black-Scholes Formula with a Continuous Dividend Yield

$$C = S^* \mathrm{N}\left(d_1\right) - Xe^{-rT}\mathrm{N}\left(d_2\right)$$

where

$$S^* = Se^{-dT}$$

$$d_1 = \frac{ln\left(S^*/X\right) + \left(r + 0.5\sigma^2\right)T}{\sigma\sqrt{T}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}$$

The following example demonstrates how to use the adjusted Black-Scholes formula to value an option written on a dividend-paying stock. Consider the two-year call option written on the common equity of the Continuous Media Corporation

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[g] See R. Merton, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science* 4 (Spring 1973): 141–183.

[h] An alternative approach is to calculate the present value of the dividends expected to be paid over the life of the option contract from the current stock price and use this adjusted stock price in the standard Black-Scholes equation. Although seemingly straightforward, this approach requires that the expected dividend stream be discounted at an appropriate rate that accounts for the riskiness of the dividend stream.

from the previous section. Assume that Continuous Media announces that, going forward, it will make regular dividend payments equivalent to a 2% continuous dividend yield. In addition, assume that Continuous Media's stock price remains unchanged at $40 following the announcement. To determine the new Black-Scholes value of the option, we first calculate the adjusted stock price S^*, calculate new values for d_1 and d_2, and construct a new replicating portfolio. The adjusted stock price is given by

$$S^* = Se^{-dT} = \$40 \times e^{-0.02\times 2} = \$38.43$$

The value of d_1 and $\mathrm{N}(d_1)$ are now given by

$$d_1 = \frac{ln\left(S^*/X\right)+\left(r+0.5\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{ln\left(\$38.43/\$35\right)+\left(0.03+0.5\times 0.4^2\right)\times 2}{0.4\times\sqrt{2}} = 0.554$$

and

$$\mathrm{N}\left(d_1\right) = \mathrm{N}\left(0.554\right) = 0.710$$

Similarly, the corresponding values of d_2 and $\mathrm{N}(d_2)$ are given by

$$d_2 = d_1 - \sigma\sqrt{T} = 0.554 - 0.4\times\sqrt{2} = -0.011$$

and

$$\mathrm{N}\left(d_2\right) = \mathrm{N}\left(-0.011\right) = 0.495$$

Given the values of $\mathrm{N}(d_1)$ and $\mathrm{N}(d_2)$, the Black-Scholes values of the call and put options are given by

Black-Scholes Call and Put Option Values

$$C = S^*\mathrm{N}\left(d_1\right) - Xe^{-rT}\mathrm{N}\left(d_2\right) = \$38.43\times 0.710 - \$35\times e^{-0.03\times 2}\times 0.495 = \mathbf{\$10.97}$$

and

$$P = Xe^{-rT} - S^* + C = \$35\times e^{-0.03\times 2} - \$38.43 + \$10.97 = \mathbf{\$5.50}$$

This example shows that dividend payments decrease the value of the call option and increase the value of the corresponding put option. **Exhibit 13** summarizes the adjusted Black-Scholes valuation assuming that Continuous Media adopts a policy of paying a continuous 2% dividend yield.

EXHIBIT 13 Black-Scholes Value with a 2% Dividend Yield

Input Assumptions		**Intermediate Calculations**	
S	$40.00	S^*	$38.43
X	$35.00	PV(X)	$32.96
T	2.00	d_1	0.554
σ	40.0%	d_2	−0.011
r_f	3.00%	N(d_1)	0.710
d	2.00%	N(d_2)	0.495

PANEL A

Calculation of Black-Scholes Call Option Value

Adjusted stock price (S^*)			$38.43
Option delta (Δ)	[= N(d_1)]	×	0.710
ΔS^*			$27.30
+PV(B)	[= −N(d_2) × PV(X)]		($16.33)
= Black-Scholes Call Option Value (*C*)			**$10.97**

PANEL B

Calculation of Black-Scholes Put Option Value

PV(X)	$32.96
+ *B-S* Call Option Value	$10.97
– Adjusted stock price	($38.43)
= Black-Scholes Put Option Value (*P*)	**$5.50**

Summary of the Black-Scholes Option Valuation Formula

In the previous sections, we have developed the techniques used to determine the value of an option contract. Specifically, we have derived the value of an option contract as the cost of constructing the portfolio that replicates the payoffs to the option contract. Because the value of an option is dependent on the price of the underlying asset on which it is written, the replicating portfolio must be continuously adjusted as the price of the underlying asset changes.

The Black-Scholes formula provides the components of the replicating portfolio—and thus the value of the option—when the price of the underlying asset changes continuously over the life of the option contract. As typically written, the Black-Scholes formula provides the value of a European call option on a non-dividend-paying stock and takes the general form:

$$C = S\mathrm{N}\left(d_1\right) - Xe^{-rT}\mathrm{N}\left(d_2\right)$$

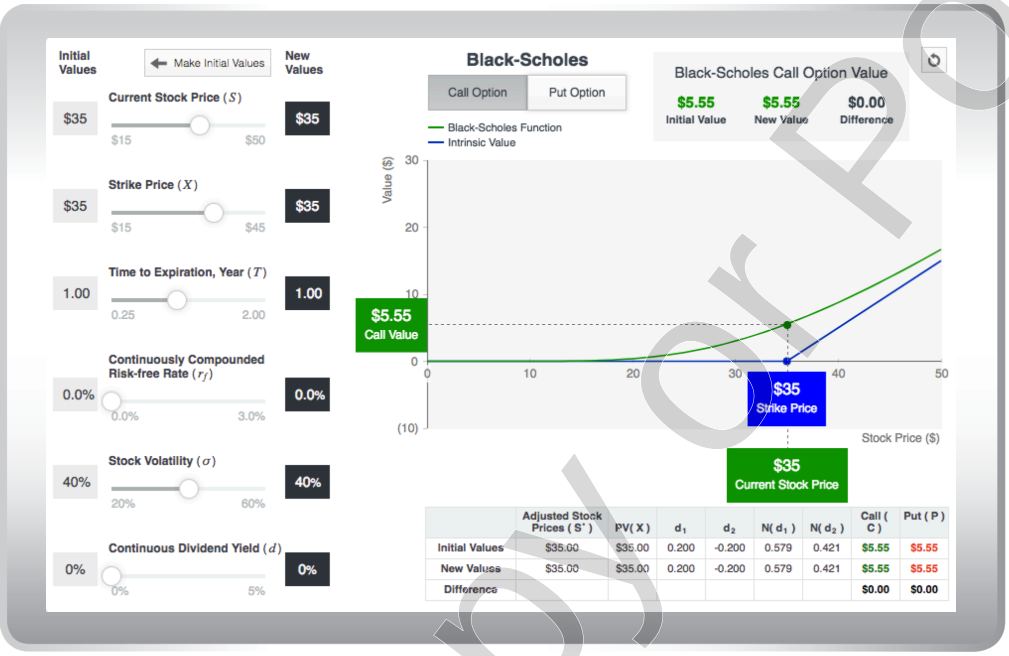
Where $\mathrm{N}(d_1)$ and $\mathrm{N}(d_2)$ are the weights used to construct the replicating portfolio. The put-call parity relationship is then used to determine the value of the corresponding European put option. Finally, the formula can be adjusted to account for the payments of dividends over the life of the contract.

In our analysis of the Black-Scholes formula, we have identified six parameter inputs that determine the value of an option: the stock price, the strike price, the time to expiration of the contract, the volatility of the underlying stock price, the risk-free interest rate, and the expected dividend yield of the underlying stock.

Interactive Illustration 7 allows you to calculate an option's value based on the Black-Scholes formula. By changing one input parameter at a time, you can see that parameter's effect on the option value, numerically and graphically, and compare initial values to new values for each of the intermediate calculations. The interactive also allows you to reset the initial values. Clicking "Make Initial Values" resets all of the initial values to the current set of new values. Then you can change parameters again and observe their effect compared with the new set of initial values.

INTERACTIVE ILLUSTRATION 7 Parameter Sensitivity of Black-Scholes Values

Scan this QR code, click the image, or use this link to access the interactive illustration: bit.ly/hbsp2ITFAV2



Initial Values		New Values
$35	Current Stock Price (S) ($15 – $50)	$35
$35	Strike Price (X) ($15 – $45)	$35
1.00	Time to Expiration, Year (T) (0.25 – 2.00)	1.00
0.0%	Continuously Compounded Risk-free Rate (r_f) (0.0% – 3.0%)	0.0%
40%	Stock Volatility (σ) (20% – 60%)	40%
0%	Continuous Dividend Yield (d) (0% – 5%)	0%

Black-Scholes Call Option Value

$5.55	$5.55	$0.00
Initial Value	New Value	Difference

	Adjusted Stock Prices (S^*)	PV(X)	d_1	d_2	N(d_1)	N(d_2)	Call (C)	Put (P)
Initial Values	$35.00	$35.00	0.200	-0.200	0.579	0.421	$5.55	$5.55
New Values	$35.00	$35.00	0.200	-0.200	0.579	0.421	$5.55	$5.55
Difference							$0.00	$0.00

3 SUPPLEMENTAL READING

3.1 Corporate Finance Applications

3.1.1 Accounting for the Cost of an Employee Stock Option Grant

To align the incentives of employees with those of shareholders, economize on cash compensation, or to encourage employee retention, corporations often include equity-linked securities as part of their employee compensation plans. *Employee stock options* (ESOs) are one such example. Financial disclosure rules require corporations to account for the cost of the ESO that they issue. The Black-Scholes option pricing formula is one method of accounting for the cost to the corporation of the ESOs that have been issued to employees. In this section, we will describe how a company might use the Black-Scholes option pricing formula to determine the cost of issuing an employee stock option.[i]

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[i] For a variety of reasons, the assumptions made in the Black-Scholes model are unlikely to hold for many employees. For example, because ESOs are generally nontransferable (and therefore non-tradable), an undiversified employee

In many ways, an employee stock option is similar to the call option contracts discussed in the previous sections. However, ESOs differ from exchange-traded options in several important ways. Similar to an exchange-traded option, an ESO contract typically gives the employee the right to purchase the company's stock at (or before) a future date at a specific price. However, with an ESO contract, the employer is the writer (seller) of the contract. This means that the number of shares outstanding will increase if the employee exercises the ESO, which has the potential to impact the price of the underlying stock.[j] In this respect, an ESO contract is similar to a *warrant* issued by the company. In addition, an ESO is a contract between the employer and the employee and, for this reason, is *nontransferable*. This means that the employee may not sell the contract to another party. If the employee wishes to convert the ESO into cash, the ESO must be exercised. For this reason, employees frequently exercise ESOs early—even though they might realize greater value if they were permitted to sell the ESO to another party. Finally, ESO contracts typically have *vesting schedules*. Although an employee may have received an *option grant*, the employee does not actually own the options until they have *vested*. As a result, an employee may not exercise an option until it has vested. In addition, any unvested ESO will be forfeited if the employee leaves the firm. For this reason, employee retention is also an important rationale for issuing ESOs. Employee stock options are typically issued with vesting schedules that extend between one and four years and expiration dates that extend between 5 and 10 years (as compared with expiration dates of generally less than a year for exchange-traded options). Finally, ESOs are generally issued at-the-money, although this is not always the case.

To see how the Black-Scholes formula can be used to determine the cost of an employee stock option grant, consider the following example. Assume you are a manager at CleanPower Energy, Inc., and you plan to give each of your employees a year-end bonus. The bonus will be structured as a grant of 500 employee stock options for each employee. You plan to use the Black-Scholes formula to determine the cost of the ESO grant. CleanPower has traditionally issued seven-year employee stock options, although its employees have on average exercised their options after 5.5 years (about 1.5 years before the expiration date). The options would vest after four years. Each year, about 5% of CleanPower's employees terminate their employment with the company, so four years from now, 18.5% of CleanPower's employees would have forfeited the options they receive this year. CleanPower's current stock price is

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will place a lower value on an ESO than implied by the Black-Scholes model. For this reason, the value of ESOs to employees is likely to be less than the cost to the firm of providing them. (See L. Meulbroek, "The Efficiency of Equity-Linked Compensation: Understanding the Full Cost of Awarding Executive Stock Options," *Financial Management* 30, no. 2 [2001]: 5–30.)

[j] Incorporating the effect on shares outstanding would be important for an employee whose ESO holdings would cause a non-negligible change in the number of shares outstanding that was not previously incorporated into the company's share price.

$50 and you plan to issue the options at-the-money with a strike price of $50. The current yields on five-year and seven-year US Treasuries are 1.82% and 2.02%, respectively. CleanPower's stock price volatility has historically been about 45% per year. Recently, however, its stock has been somewhat more volatile; in fact, volatility implied from the market prices of CleanPower's exchange-traded options has been around 60%. CleanPower does not expect to pay dividends in the foreseeable future, so its expected dividend yield is zero. You want to construct a reasonable range for the cost of your planned year-end bonus.

Because we are not certain when employees may exercise their options nor certain how volatile the CleanPower stock price may be over the life of the ESO grant, we can calculate the Black-Scholes value of the ESO grant under several different scenarios. For example, we can calculate the value of the option grant using both CleanPower's historical stock price volatility as well as the implied volatility from its exchange-traded options. In addition, we can calculate the value of the option grant using the average time at which CleanPower employees exercise their ESO grants as well as the time to expiration of the ESO grant. **Exhibit 14** summarizes these four scenarios. Scenarios 1 and 2 use the historic average duration of 5.5 years, and Scenarios 3 and 4 use the full contract duration, seven years. Scenarios 1 and 3 use the lower historic volatility of 45%, and Scenarios 2 and 4 use the recent and higher volatility of 60%.

EXHIBIT 14 Expected Cost of the Planned CleanPower ESO Grant

	Exercise Early		**Exercise at Expiration**	
Scenario:	**1**	**2**	**3**	**4**
Years to exercise	5.5	5.5	7.0	7.0
Volatility	45%	60%	45%	60%
Risk-free rate	1.82%	1.82%	2.02%	2.02%
B-S value per option	$21.62	$27.11	$24.37	$30.13
Options per grant	500	500	500	500
Grant value	**$10,808**	**$13,557**	**$12,183**	**$15,063**
Forfeiture probability	18.5%	18.5%	18.5%	18.5%
Expected grant cost	**$8,809**	**$11,049**	**$9,929**	**$12,276**

The above analysis shows that the planned year-end bonus is expected to cost CleanPower between $8,809 and $12,276 per employee.

3.1.2 Valuing the Terms of an Acquisition Offer

The financing of mergers and acquisitions (M&A) is an important element of corporate finance. One aspect of M&A activity is the type of consideration used to finance transactions. In addition to the value of equity included in a particular offer, additional terms and conditions are often specified. As a shareholder of either the acquirer or the target, understanding how to assess and value the specific terms of an acquisition offer is crucial to determining whether you think the deal terms are favorable, need to be revised, or rejected outright.[k] In this section, we will show how the Black-Scholes option pricing formula can be used to value the terms of an acquisition offer.

When an acquirer makes an offer in an equity deal, the acquirer typically offers to exchange some number of its shares for each target share that it acquires. This ratio is referred to as the *exchange ratio*. For example, assume that EmpireCo has made an all-equity offer for CleanPower Energy, Inc. EmpireCo has offered to exchange two of its shares for each outstanding CleanPower share. EmpireCo's current stock price is \$30. Thus, based on the current value of EmpireCo's shares, it is offering to pay \$60 per share to acquire CleanPower. Given that CleanPower's current share price is \$50, the EmpireCo offer constitutes a 20% premium over the current CleanPower share price. However, it is expected to take approximately six months to complete the transaction. Because the future price of EmpireCo shares is uncertain (the volatility of EmpireCo's stock price is 35% per year), CleanPower shareholders are not guaranteed to receive \$60 per share when the transaction actually closes. The uncertainty in the value of the consideration to be paid at closing is a potential reason for CleanPower shareholders to be uncomfortable with EmpireCo's offer. Anticipating these potential concerns, EmpireCo has also stipulated that it will deliver consideration worth *at least* \$48 per share at closing. To make an informed decision whether to accept or reject the offer, you want to know the current value of the stated EmpireCo offer.

To determine the current value, consider the corresponding payoff function implied by the offer's terms. Because EmpireCo is guaranteeing a minimum payment of \$48 per share, the payoff function has two parts: (1) When the EmpireCo share price is greater than or equal to \$24, CleanPower shareholders will receive two EmpireCo shares; and (2) When the EmpireCo share price is less than \$24, CleanPower shareholders will receive a number of EmpireCo shares with value equaling \$48. **Exhibit 15** shows the payoff function implied by the EmpireCo offer.

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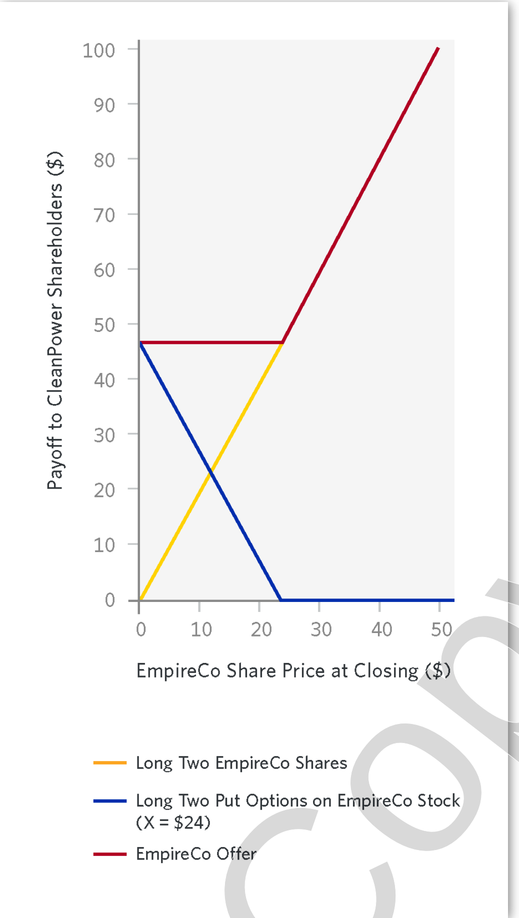
[k] For more information about mergers and acquisitions, refer to *Core Reading: Mergers and Acquisitions* (HBP No. 5242).

EXHIBIT 15 Payoff Function Implied by the EmpireCo Offer



From the above payoff function, we can see that the value of the EmpireCo offer is equivalent to the value of two EmpireCo shares plus two put options written on EmpireCo stock, each with a strike price of $24. For this reason, we need to value the embedded put options to determine the current value of the EmpireCo offer. **Exhibit 16** shows the calculation of the Black-Scholes value of the implied EmpireCo put option.

EXHIBIT 16 Black-Scholes Value of the Embedded EmpireCo Put Option

Input Assumptions		**Intermediate Calculations**	
S	\$30.00	S^*	\$30.00
X	\$24.00	PV(X)	\$23.94
T	0.50	d_1	1.035
σ	35.0%	d_2	0.788
r_f	0.50%	N(d_1)	0.850
d	0.00%	N(d_2)	0.785

PANEL A
Calculation of Black-Scholes Call Option Value

Adjusted stock price (S^*)			\$30.00
Option delta (Δ)	[= N(d_1)]	×	0.850
ΔS^*			\$25.49
+PV(B)	[= –N(d_2) × PV(X)]		(\$18.78)
= Black-Scholes Call Option Value (*C*)			**\$6.71**

PANEL B
Calculation of Black-Scholes Put Option Value

PV(X)	\$23.94
+ *B-S* Call Option Value	\$6.71
– Adjusted stock price	(\$30.00)
= Black-Scholes Put Option Value (*P*)	**\$0.65**

Given that the six-month Treasury rate is 0.5% and that EmpireCo does not pay any dividends, the exhibit above shows that a put option written on EmpireCo stock with an exercise price of \$24 is currently worth \$0.65. Thus, the current value of the EmpireCo offer equals \$61.30, the current value of two EmpireCo shares (2 × \$30) plus the current value of two put options on EmpireCo stock, struck at \$24, (2 × \$0.65).

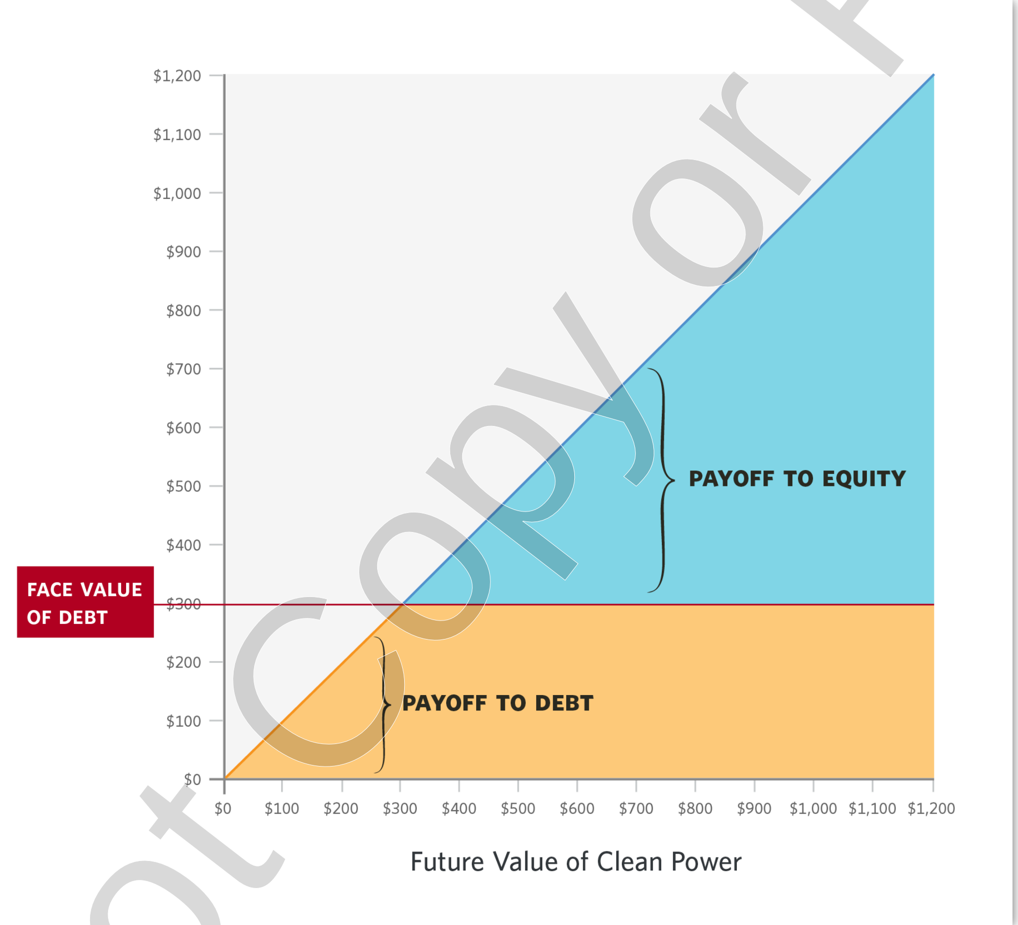
3.1.3 Determining an Appropriate Yield to Maturity on a New Debt Issue

Raising new capital is another important element of corporate finance. Firms often need to raise capital to finance new projects, cover operating losses, or adjust their existing capital structure. Whether the firm raises capital in the form of a new debt issue, a secondary equity offering, or some hybrid security, it is crucial that the security be fairly priced. In other words, the security should offer investors an expected rate of return that compensates for the risk associated with holding the security, but—if the security is fairly priced—no more and no less. In this section, we will show how our option-pricing framework can be used to determine the appropriate pricing of corporate securities.

To see how the option-pricing framework can be used to determine the pricing of corporate securities, consider the following example. Assume that you are the CFO of CleanPower Energy, Inc., and an activist investor has been campaigning for a change in CleanPower's current capital structure. Specifically, the investor has been arguing that CleanPower needs to include a significant amount of debt in its capital structure. Historically, CleanPower has used only equity to finance its operations, and as a result, currently has no debt outstanding. At present, CleanPower's total enterprise is worth \$1 billion; its share price is \$50 and it has 20 million shares outstanding. (For various reasons, including the accumulation of previous operating losses and existing investment tax credits, CleanPower will not pay corporate taxes for the foreseeable future.) The activist investor is suggesting that CleanPower should have a significant amount of debt in its capital structure and should undertake a *leveraged recap*; the specific proposal is to issue \$300 million face value of five-year zero-coupon bonds and to use the proceeds from the debt issue to repurchase a portion of the current outstanding equity. As the CleanPower CFO, you need to make a recommendation to the board regarding the proposal. However, before you can offer an opinion, you need to determine an appropriate yield to maturity on the new debt issue.

To determine an appropriate yield to maturity for the new debt issue, consider the shareholders' payoff function after the proposed recapitalization. At present, the CleanPower assets are worth \$1 billion, but the future value of the CleanPower assets is uncertain. After the recapitalization, equity will have a *residual claim* on the CleanPower assets, meaning that the equity holders have the *right* to claim ownership of the CleanPower assets by repaying the more senior \$300 million debt claim. If the future value of the CleanPower assets exceeds \$300 million, it will be rational for the equity holders to repay the debt claim. However, if the future value of the CleanPower assets is less than \$300 million, equity holders are *not obligated* to claim ownership of the CleanPower assets. Instead, they can forfeit their ownership of the CleanPower assets to the debt holders. In other words, the CleanPower equity claim is equivalent to a *call option* written on the underlying CleanPower assets. **Exhibit 17** shows the payoff function for CleanPower equity after the proposed recapitalization.

EXHIBIT 17 Payoffs to Debt and Equity Based on the Future Value of CleanPower



Conversely, the holders of the new debt will have a *risky* claim that may not be repaid in full. Because of the risk of default, the new debt issue must offer a yield to maturity that compensates for this risk. For this reason, we can use the Black-Scholes formula to determine the post-recapitalization values of the equity and debt claims and thus the required yield to maturity on the new debt issue.

We will determine an appropriate yield to maturity of the new debt issue by determining the value of the debt when issued and using that value to calculate the debt's yield to maturity. After the proposed recapitalization, the value of the CleanPower debt will be given by the identity

$$\text{Value}\left(\text{CleanPower Debt}\right) = \text{Value}\left(\text{CleanPower Assets}\right) - \text{Value}\left(\text{CleanPower Equity}\right)$$

Assuming that the proposed recapitalization does not change the value of the underlying CleanPower assets, we can rewrite the expression to yield the value of the new CleanPower debt, such that

$$\text{Value}\left(\text{CleanPower Debt}\right) = \$1{,}000 \text{ million} - C$$

where *C* is the value of the post-recapitalization CleanPower equity claim. After the proposed recapitalization, the value of the CleanPower equity claim will be equal to the value of a five-year call option written on the underlying CleanPower assets with a strike price of $300 million. The underlying CleanPower assets are valued at $1 billion and have an annual volatility of 45% per year. (Before the recap, CleanPower had no debt, so the historical volatility of its equity reflects the historical volatility of its assets.) Given these assumptions, **Exhibit 18** shows the calculation of the Black-Scholes value of the CleanPower equity claim.

EXHIBIT 18 Black-Scholes Value of the CleanPower Equity Claim (Dollar Values in Millions)

Input Assumptions		**Intermediate Calculations**	
S	\$1,000	S^*	\$1,000
X	\$300	PV(X)	\$274
T	5.00	d_1	1.790
σ	45.0%	d_2	0.784
r_f	1.82%	N(d_1)	0.963
d	0.00%	N(d_2)	0.783

PANEL A
Calculation of Black-Scholes Call Option Value

Adjusted stock price (S^*)		\$1,000
Option delta (Δ)	[= N(d_1)]	× 0.963
ΔS^*		\$963
+PV(B)	[= −N(d_2) × PV(X)]	(\$215)
= Black-Scholes Call Option Value (C)		**\$749**

PANEL B
Calculation of Black-Scholes Put Option Value

PV(X)	\$274
+ *B-S* Call Option Value	\$749
– Adjusted stock price	(\$1,000)
= Black-Scholes Put Option Value (P)	**\$23**

The above exhibit shows that, given the terms of the proposed recapitalization, the Black-Scholes value of the CleanPower equity claim equals \$749 million. Given the value of the equity claim, the value of the CleanPower debt claim must equal \$251 million (= \$1,000 million – \$749 million). Finally, given the value of the proposed CleanPower debt, the implied yield to maturity (YTM) for the new debt is given by the formula

$$\text{YTM} = \left(\frac{\$300}{\$251}\right)^{1/5} - 1 = 3.6\%$$

This shows that the required yield to maturity for the proposed CleanPower debt equals 3.6%.

We can also derive the yield to maturity for the new debt by using the value of the corresponding put option. Given put-call parity, the present value of the risky CleanPower debt is equal to the present value of a risk-free $300 million face value bond minus the value of a put (P) written on the underlying CleanPower assets.

With this formulation, the value of the new CleanPower debt is given by

$$\text{Value}\left(\text{CleanPower Debt}\right) = \$300e^{-rT} - P = \$300e^{-1.82\% \times 5} - \$23 = \$251$$

Because this second approach produces exactly the same value for the new CleanPower debt as our first approach, we get the same required yield to maturity of 3.6%.

3.2 Understanding the Early Exercise Decision

So far, we have focused primarily on *European* options—options that can be exercised only at expiration. In contrast, *American* options can be exercised at any time during the life of the option contract. However, it is important to understand that although a holder of an American option has the right to exercise the option before expiration, it may not be rational to do so. In many instances, it will be better to either hold the option until expiration or to sell it to someone else who will hold the option until expiration. In this section, we will analyze the decision to exercise an American option before expiration and the resulting implications for the value of an American option relative to its European counterpart.

Because an American option may be exercised anytime during the life of the contract, it is more flexible than its European counterpart. In other words, the holder of an American option can always choose to treat the option as a European option, but the holder of a European option cannot choose to treat the option as an American option. For this reason, the value of an American option can never be less than the value of its European counterpart. Therefore, the following relationships must hold:

$$C_A \geq C_E \geq 0$$

and

$$P_A \geq P_E \geq 0$$

where C_A and P_A are the values of American call and put options, respectively, and C_E and P_E are the values of the corresponding European call and put options. From this inequality, we know that if it is optimal to exercise an American option before expiration, then the value of the American option must be greater than the value of its European counterpart. Conversely, if it is not optimal to exercise an American option before expiration, then the value of the American option will equal the value of its European counterpart. For these reasons, it is important to understand the factors that determine the decision to exercise an option before its expiration date.

We will structure the analysis of the early exercise decision around the observation that an American option will only be exercised early if its *option value* is less than its *intrinsic value*. Otherwise, the early exercise of an American option would destroy value. Specifically, it will be optimal to exercise an American call option early *only if*

$$C_A < S - X \tag{1}$$

Similarly, it will be optimal to exercise an American put option early *only if*

$$P_A < X - S$$

3.2.1 Early Exercise of American Call Options

We will use the put-call parity relationship for options written on dividend-paying and non-dividend-paying stocks to explain when it may be optimal to exercise an option before the expiration date.

First, consider a European call option written on a non-dividend-paying stock. Put-call parity requires that

$$C_E + \text{PV}(X) = P_E + S$$

where X is the option's strike price, $\text{PV}(X)$ is the present value of the strike price, and S is the current stock price.

However, because value of the put is always non-negative ($P \geq 0$), we can rewrite the above expression as

$$C_E = P_E + S - \text{PV}(X) \geq S - \text{PV}(X)$$

In addition, because $\text{PV}(X) \leq X$, we also know that

$$C_E \geq S - \text{PV}(X) \geq S - X$$

Finally, we know that value of an American option must be no less than the value of its European counterpart, such that

$$C_A \geq C_E \geq S - X$$

This last relationship shows that the value of an American call option written on a non-dividend-paying stock is *always greater than or equal to* its payoff if exercised (its intrinsic value), $max(S - X, 0)$. Recall from equation (1) that an American call option will be exercised only if its value is *less than* its intrinsic value. For this reason, it is never optimal to exercise an American call option written on a non-dividend-paying stock before expiration.

Now, consider a European call option written on a dividend-paying stock. Remember that the price of a dividend-paying stock equals the present value of the stock's expected payoff at the expiration date plus the present value of the dividends expected to be paid during the life of the contract, such that

$$S = \text{PV}(\text{Dividends}) + \text{PV}\left(\text{Expected}\left[S_T\right]\right)$$

For this reason, put-call parity requires that

$$C_E + \text{PV}(X) + \text{PV}(\text{Dividends}) = P_E + S$$

Because $P \geq 0$, we can rewrite the above expression as

$$C_E = P_E + S - \text{PV}(\text{Dividends}) - \text{PV}(X) \geq S - \text{PV}(\text{Dividends}) - \text{PV}(X)$$

Finally, because $C_A \geq C_E$ and $\text{PV}(X) \leq X$, we have

$$C_A \geq \left[S - \text{PV}(\text{Dividends})\right] - X$$

However, if PV(Dividends) > 0, then

$$C_A \geq \left[S - \text{PV}(\text{Dividends})\right] - X < S - X$$

This last relationship shows that the value of an American option written on a dividend-paying stock, C_A, can be less than the intrinsic value of the option, $max(S - X, 0)$. As a result, when an American option gets close to expiration and the time value associated with holding the option is less than the present value of the remaining expected dividend payments, it will be optimal to exercise the option before the expiration date. In other words, because the price of the underlying stock falls when a dividend is paid, the loss in value associated with holding the option past the ex-dividend date will exceed the loss in value associated with exercising the option early. For this reason, it may be optimal to exercise an American option written on a dividend-paying stock before the expiration of the option.

3.2.2 Early Exercise of American Put Options

Similar to the above analysis, we will again use the put-call parity relationship to understand when it will be optimal to exercise an American put option before the expiration date. First, consider the put-call parity relationship for a European put option written on a non-dividend-paying stock. Put-call parity requires that:

$$C_E + \mathrm{PV}(X) = P_E + S$$

Because $C_E \geq 0$, we can rewrite the above expression, such that

$$P_E = C_E + \mathrm{PV}(X) - S \geq \mathrm{PV}(X) - S$$

In addition, because $P_A \geq P_E$, we know that

$$P_A \geq P_E \geq \mathrm{PV}(X) - S$$

Finally, we can rewrite the right-hand side of the above expression as

$$P_A \geq \left[\mathrm{PV}(X) - X\right] + X - S$$

This last expression shows that $P_A \geq X - S$ if

$$\mathrm{PV}(X) - X \geq 0$$

If the risk-free interest rate is zero, then $\mathrm{PV}(X) = X$ and it will not be optimal to exercise an American put option before the expiration date. However, if the risk-free interest rate is positive (which is generally the case), then $\mathrm{PV}(X) < X$ and it may be optimal to exercise an American put option early. Specifically, because a put option gives the holder the right to sell the underlying stock in the future, when the option is sufficiently in-the-money—such that there is little option value associated with holding the option—it may be optimal to exercise the option before the expiration date. In other words, the loss in value associated with exercising the option early may be less than the additional interest earned by receiving the strike price before the expiration date of the option.

Finally, consider a European put option written on a dividend-paying stock. Because the price of the underlying stock reflects the present value of the stock's expected payoff at the expiration date plus the present value of the expected dividends, put-call parity requires that

$$C_E + \mathrm{PV}(X) + \mathrm{PV}(\mathrm{Dividends}) = P_E + S$$

Because $C_E \geq 0$, we can rewrite the above expression as

$$P_E = C_E + \mathrm{PV}(X) + \mathrm{PV}(\mathrm{Dividends}) - S \geq \mathrm{PV}(X) + \mathrm{PV}(\mathrm{Dividends}) - S$$

In addition, because $P_A \geq P_E$, we know that

$$P_A \geq P_E \geq \mathrm{PV}(\mathrm{Dividends}) + \mathrm{PV}(X) - S$$

Finally, we can rewrite the right-hand side of the above expression, such that

$$P_A \geq \left[\mathrm{PV}(\mathrm{Dividends}) + \mathrm{PV}(X) - X\right] + X - S$$

This last expression shows that $P_A \geq X - S$ if

$$\mathrm{PV}(\mathrm{Dividends}) \geq X - \mathrm{PV}(X)$$

In other words, if the present value of the remaining expected dividends exceeds the interest rate effect from early exercise, then it will be optimal to hold the put option until the expiration date.

3.3 Calculating an Option's Value from Expected Future Payoffs

In the Essential Reading section, we used the concept of a replicating portfolio and the law of one price to determine the value of an option contract. In this section, we demonstrate a technique for determining the value of an option as the present discounted value of its expected future cash flows.

We start with the value of an option as equal to the value of the portfolio that replicates its payoffs,

$$C = \Delta S + \text{PV}(B)$$

Where $B < 0$. Rearranging the above equation, we can construct a portfolio that has a constant, *guaranteed* future payoff of $G = -B$, regardless of the future stock price. The portfolio contains Δ shares of stock and a short call:

Portfolio Components

$$\Delta S - C$$

We use the discrete two-payout example to calculate expected payoffs to the portfolio at expiration, T, of the call contract:

Portfolio Payoffs

$$\text{Payoff}_T = \Delta S_T - C_T = \begin{cases} \Delta S_H - C_H = G & \text{if } S_T = S_H \\ \Delta S_L - C_L = G & \text{if } S_T = S_L \end{cases}$$

Because the payoff to the portfolio is guaranteed and thus has no risk, we can discount the portfolio payoffs to the present at the risk-free interest rate (r_f). In addition, because the payoffs are constant, a linear combination of any two payoffs results in the same payoff. Thus, the following relationships must hold:

$$\Delta S - C = \frac{G}{1+r_f} = \left(\frac{\pi G + (1-\pi)G}{1+r_f}\right) = \left(\frac{\pi(\Delta S_H - C_H) + (1-\pi)(\Delta S_L - C_L)}{1+r_f}\right)$$

where π and $(1 - \pi)$ are the weights given to the two payoffs. Rearranging the above formula provides a description of the portfolio in terms of the future stock and call payoffs:

$$\Delta S - C = \Delta\left(\frac{\pi S_H + (1-\pi)S_L}{1+r_f}\right) - \left(\frac{\pi C_H + (1-\pi)C_L}{1+r_f}\right)$$

Rearranging further so that all of the stock price terms are on the left-hand side of the equation and all of the call option terms are on the right-hand side of the equation gives the following expression:

$$\Delta\left[S-\left(\frac{\pi S_H+(1-\pi)S_L}{1+r_f}\right)\right]=C-\left(\frac{\pi C_H+(1-\pi)C_L}{1+r_f}\right) \tag{2}$$

We know that one solution of equation (2) is where both the left- and right-hand sides equal zero. We determine the value of π (call it π^*) at which this occurs, such that

$$S=\frac{\pi^* S_H+(1-\pi^*)S_L}{1+r_f} \tag{3}$$

and

$$C=\frac{\pi^* C_H+(1-\pi^*)C_L}{1+r_f} \tag{4}$$

These equations suggest a simple way to determine the value of an option from its expected future payoffs: First, using equation (3), find the value of π^* such that the weighted average of future stock payoffs, discounted at the risk-free interest rate r_f, equals the current stock price, *S*:

$$\pi^*=\frac{S(1+r_f)-S_L}{S_H-S_L}$$

Then, to find the present value of the call option, use π^* in equation (4) to calculate the corresponding weighted average of the future call option payoffs, and discount that weighted average to the present at the risk-free interest rate r_f.

To see that this method for valuing a call option is consistent with our method based on constructing a replicating portfolio, we can use it to value the Simple Solutions call option from the discussion of replicating portfolios in section 2.2.1. In the example of Simple Solutions, the current stock price was \$35, the future stock price was either \$60 or \$20, the call contract strike price was \$40, and the risk-free rate was zero. Therefore the value of π^* is

$$\pi^*=\frac{\$35\times(1+0\%)-\$20}{\$60-\$20}=\frac{\$15}{\$40}=0.375$$

Substituting this value into the expression for the call option yields the same option value we obtained using the replication method:

$$C=\frac{0.375\times\$20+\left(1-0.375\right)\times\$0}{1+0\%}=0.375\times\$20=\$7.5$$

The fact that the value of an option contract can be thought of as the present discounted value of its expected payoff allows for an alternative interpretation of the Black-Scholes option pricing formula. The Black-Scholes formula (discussed in section 2.2.3) is given as

$$C=S\mathrm{N}\left(d_1\right)-Xe^{-rT}\mathrm{N}\left(d_2\right)$$

We can rewrite the above formula as

$$C=\left[\mathrm{N}\left(d_2\right)\left(Se^{rT}\frac{\mathrm{N}\left(d_1\right)}{\mathrm{N}\left(d_2\right)}-X\right)\right]e^{-rT}$$

or equivalently, as

$$C=\left[\mathrm{N}\left(d_2\right)\times\left(Se^{rT}\frac{\mathrm{N}\left(d_1\right)}{\mathrm{N}\left(d_2\right)}-X\right)+\left(1-\mathrm{N}\left(d_2\right)\right)\times 0\right]e^{-rT}$$

The above equation can be interpreted as

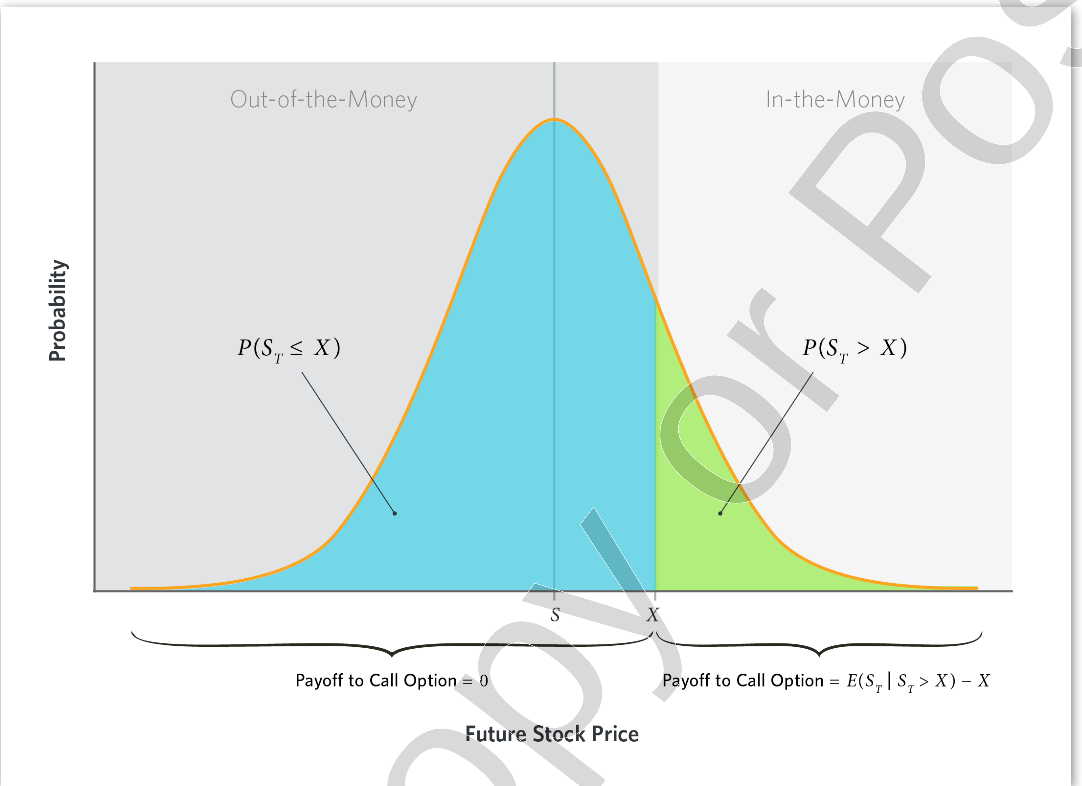
$$C=\frac{\mathrm{P}\left(S_T>X\right)\times\left(\mathrm{E}\left[S_T\middle|S_T>X\right]-X\right)+\left(1-\mathrm{P}\left(S_T>X\right)\right)\times 0}{e^{rT}}$$

Where $\mathrm{N}(d_2)$ is the probability that the call option will be in-the-money at expiration, $\mathrm{P}(S_T > X)$; and

$$Se^{rT}\frac{\mathrm{N}\left(d_1\right)}{\mathrm{N}\left(d_2\right)}$$

is the expected value of S_T, conditional on S_T being greater than X (the green area depicted in **Exhibit 19**).

EXHIBIT 19 Option Value and Expected Future Payoff



Finally, we can simplify the above expression as

$$C = \frac{\mathrm{P}\left(S_T > X\right) \times \left(\mathrm{E}\left[S_T \middle| S_T > X\right] - X\right)}{e^{rT}}$$

This final expression calculates the value of a call option as the present discounted value of the option's expected future payoff, where the future payoff is discounted to the present at the risk-free rate and the probability of the option being in-the-money is calculated "as if" the expected return on the underlying stock were equal to the risk-free rate.

4 FURTHER READING

Cox, John C., and Mark Rubinstein, *Options Markets*, 1st edition. Boston: Prentice-Hall, (1985).

Hull, John C., *Options, Futures, and Other Derivatives*, 9th edition. Boston: Prentice-Hall, (2014).

5 KEY TERMS

American option	An option that can be exercised at any time before or on its expiration date.
at-the-money	Describes an option whose strike price is equal to the current value of the underlying asset.
call option	A contract that gives the owner the right, but not the obligation, to purchase an underlying asset for a specified price at or before a specified future date.
cumulative standard normal distribution	A normal distribution with mean of 0 and standard deviation of 1. Values range from 0 to 1.
European option	An option that can be exercised only on its expiration date.
exercise (an option)	An option contract gives its owner the right but not the obligation to buy or sell an underlying asset at a specified price; when the owner chooses to go ahead with that purchase or sale, he or she is exercising the option.
exercise price	The price, specified in an option contract, at which the option owner can buy (for a call option) or sell (for a put option) the underlying asset; also called *strike price.*
expiration date (for an option)	The date at which an option contract expires. A European option may be exercised only on the expiration date. An American option can be exercised before or on the expiration date.
financial distress	Occurs when a firm has difficulty paying or is unable to pay its creditors.

in-the-money Describes an option whose intrinsic value is positive. Call options are in-the-money when the current price of the underlying asset is higher than the option's strike price. Put options are in-the-money when the current price of the underlying asset is lower than the option's strike price.

intrinsic value (of an option) The value of an option if it were exercised at the current price of the underlying asset.

law of one price Stipulates that two assets with identical payoffs must have the same price.

leverage Refers to the amount of debt used to finance an asset. Higher leverage indicates a higher percentage of debt relative to equity.

long position Refers to owning an asset.

option buyer or option holder Refers to the person or institution who has purchased an option contract. The buyer has the right to exercise the option.

option delta Measures the change in an option's value with respect to a change in the value of the underlying asset.

option seller or option writer Refers to a person or institution who has sold an option contract. The seller sells exercise rights to the buyer.

out-of-the-money Describes an option whose intrinsic value is negative. Call options are out-of-the-money when the current price of the underlying asset is lower than the option's strike price. Put options are out-of-the-money when the current price of the underlying asset is higher than the option's strike price.

payoff diagram A diagram that shows the payoff to an option (on the vertical axis) based on the future value of the underlying asset (on the horizontal axis)

payoff function A formula that defines the payoff to an option based on the future value of the underlying asset.

premium (for an option) The price of an option contract in excess of its intrinsic value.

put option A contract that gives the owner the right, but not the obligation, to purchase an underlying asset for a specified price at or before a specified future date.

put-call parity Defines the relationship between the value of a European call option and the associated European put option.

replicating portfolio A portfolio of assets whose payoff is exactly the same as a given asset.

short position Refers to an asset that is owed rather than owned.

strike price Specified in an option contract, the price at which the option owner can buy (for a call option) or sell (for a put option) the underlying asset; also called *exercise price*.

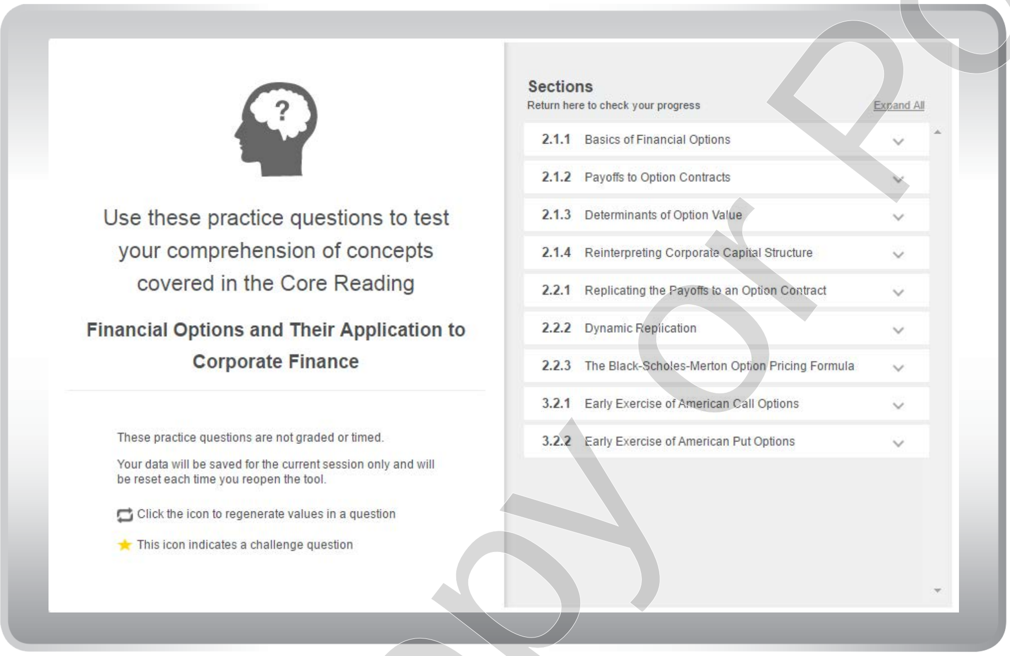
zero-coupon bond A bond that pays no coupons. The entire face value is repaid at maturity.

6 NOTATION

C	value of a call option
C_H	call option value when price of underlying asset is high
C_L	call option value when price of underlying asset is low
C_T	value of a call option at time T
d	dividend yield
Δ	option delta
P	value of a put option
P_T	value of a put option at time T
r	discount rate
r_f	risk-free rate
S	stock price
S_H	high stock price
S_L	low stock price
S_T	stock price at time T
σ	standard deviation or volatility
T	time
X	strike price
YTM	yield to maturity (of a bond)

7 PRACTICE QUESTIONS

Scan this QR code, click the image, or use this link to access the interactive illustration: bit.ly/hbsp2pKuUA0



Use these practice questions to test your comprehension of concepts covered in the Core Reading

Financial Options and Their Application to Corporate Finance

These practice questions are not graded or timed.

Your data will be saved for the current session only and will be reset each time you reopen the tool.

Click the icon to regenerate values in a question

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Sections

Return here to check your progress

Expand All

- **2.1.1** Basics of Financial Options
- **2.1.2** Payoffs to Option Contracts
- **2.1.3** Determinants of Option Value
- **2.1.4** Reinterpreting Corporate Capital Structure
- **2.2.1** Replicating the Payoffs to an Option Contract
- **2.2.2** Dynamic Replication
- **2.2.3** The Black-Scholes-Merton Option Pricing Formula
- **3.2.1** Early Exercise of American Call Options
- **3.2.2** Early Exercise of American Put Options

8 INDEX